A Study of Markov Chain

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Abstract

Stochastic models are being used to an ever increasing extent to investigate phenomena .The processes in discrete time is Markov Chain. We dealt it in the application of Gambler's Ruin problem. Moreover, we establish the drug testing.

Key words: probability, transition, absorbing state, ergodic

Introduction

In this paper we consider the various form of Markov Chain. The stochastic process $\{ X_n , n = 0 , 1 , 2 , . . . \}$ is called a Markov Chain if for $i , k , i_1 , i_2 , . . . , i_{n-1} \in N$,

$$P\{ X_n = k \ / \ X_{n-1} = i \ , \ X_{n-2} = i_1 . . . X_0 = i_{n-1} \} = P \{ X_n = k \ / \ X_{n-1} = i \} = P_{ik} \quad ..... \quad (1.1)$$

Equation (1.1) implies that given the present state of the system, the future is independent of its past. If the state space is discrete in a Markov process, then it is called Markov Chain.

Further, if the parameter space is also discrete, then the Markov Chain is called discrete parameter Markov Chain. The values $i , i_1 , i_2 , i_3 , . . . , i_{n-1}$ are called the states of the Markov Chain.

0-step transition

If the transition probability P_{jk} is independent of n, the Markov Chain is said to be homogeneous or to have stationary transition probability. If it is dependent on n, the chain is said to be non-homogeneous (Baily, 1964).

The transition s probability P_{jk} refers to the state (j , k) at two successive trials

(n^{th} and $(n+1)^{th}$), the transition in one step and P_{jk} is called one- step transition probability .

In general , if the pair of states (j , k) at two successive trials say state j at the n^{th} trial and the state at the $(n + m)^{th}$ trial, then the corresponding transition probability is called m-step transition probability and it is denoted by $P_{jk}(m)$, where $P_{jk}(m)= P\{ X_{n+m} = k / X_{n = j} \}$.

0-step transition probabilities is given by $P_{jk}(0) = \begin{cases} 1 , \text{ if } j = k \\ 0 , \text{ otherwise .} \end{cases}$

Transition Matrix

The transition probability P_{jk} satisfy $P_{jk} \geq 0$, $\sum_k P_{jk} = 1$ for all j .The one-step transition probabilities are compactly specified in the matrix form

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$$p = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \\ . & . & . & \cdots \end{bmatrix}$$

This is called the transition probability matrix of the Markov Chain. Any such square matrix that has non-negative entries and unit row sums is called a *stochastic matrix.*

Any equivalent description of one-step transition probabilities is given by a discrete graph is called the state-transition diagram.

The probability mass function of the random variable x_0 is called the initial distribution and is specified by the probability vector P(0) = [$P_0(0)$, $P_1(0)$,…] and P(n) is called the state probability vector after n or the probability distributions of X_n .

Order of Markov Chain

A Markov Chain { X_n } is said to be of **order S** (S = 1,2,…) if for all n ,

$P [X_n = k / X_{n-1} = i_{n-1} , X_{n-2} = i_{n-2} , . . . , X_{n-s} = i_{n-s} , . . .] = P [X_n = k / X_{n-1} = i_{n-1} , X_{n-2} = i_{n-2} , . . . , X_{n-s} = i_{n-s}]$.

A Markov Chain { X_n } is said to be of **order one**

$P [X_n = k / X_{n-1} = j , X_{n-2} = j_1 , \ldots] = P [X_n = k / X_{n-1} = j] = P_{jk}$.

Irreducible State

If the probability P_{jk} (n) is non-zero for some n ≥ 1, then we say that the state k can be reached from the state j . If every state can be reached from every other state (in any number of transitions) then the chain is said to be irreducible . The transition matrix is irreducible. State j is said to be accessible from state i, if for some n ≥ 0, P_{jk} (n) > 0 and we write $i \to j$.

Absorbing State

If C is a set of states such that no state outside of C can be reached from any state of C , then the set of states C is said to be closed . If a closed set contains only one state then the state is called an absorbing state. That is, a state i is said to be absorbing if $P_{ii} = 1$, and

$P_{ik} = 0$ for $i \neq k$.

The probability that stating with j , the state k is reached for the first time at the r^{th} step and again after that at $(n-r)^{th}$ step is given by f_{jk} (r) P_{kk} (n-r) , n ≥ 1, with P_{kk} (0) = 1,

f_{jk} (r) = 0 , f_{jk} (1) = P_{jk} .

Let F_{jk} denote the probability of ever visiting state k , starting from state j . Then

$F_{jk} = \sum_{n=1}^{\infty} f_{jk}(n)$, where $f_{jk}(n) = \sum_{j \neq k} P_{ji} f_{jk}(n-1)$, n = 2, 3, ….

The mean (first passage) time from the state j to the state k is given by

$\mu_{jk} = \sum_{n=1}^{\infty} n f_{jk}(n)$, $\mu_{jj} = \sum_{n=1}^{\infty} n f_{jj}(n)$ is known as the mean recurrence time for j .

Recurrent State

A state i is said to be recurrent if and only if starting from state i , the process eventually returns to state i with probability 1. That is, if $F_{ii} = 1$ (i.e return to the state i is certain). A state i is said to be transient (or nonrecurrent) if and only if there is a positive probability that the process will not return to this state . That is the state i is transient if $F_{ii} < 1$.

A recurrent state i is said to be null if $\mu_{ii} = \infty$, that is if the mean recurrence time is infinite . A recurrent state is said to be non-null if $\mu_{ii} < \infty$. For a recurrent state i , $P_{ii}(n) > 0$ for some $n \geq 1$. The period of state i , by i > 1 which is the greatest common divisor of the set of positive integer such that $P_{ii}(n) > 0$. A recurrent state i is said to be aperiodic if no such t > 1 exists .A recurrent state i is said to be periodic if t > 1.

Aperiodic State

A recurrent non-null and aperiodic state of a Markov Chain is said to be ergodic. A Markov Chain all of whose states are ergodic is said to be an ergodic chain. Consider a

Markov Chain with transition probability P_{jk} . A probability distribution $\{ v_j \}$ is called stationary (or invariant) for the given chain if $v_k = \sum_j v_j P_{jk}$ such that $v_j \geq 0$, $\sum_j v_j = 1$. If the numbers v_j, $j \in I$ are such that $\sum_{j \in I} v_j = 1$, then v_j's are said to form a stead- state distribution.

Regular Chain

A Markov Chain is called regular if there is a finite positive integer m such that after m time-steps, every state has a non-zero chance of being occupied .

A Markov Chain is absorbing if (1) it has at least one absorbing state, (2) it is possible to go from every non-absorbing in one step .

Lemma

Let $\{ f_n \}$be a sequence such that $f_0 = 0$, $f_n \geq 0$ and $\sum_{n=1}^{\infty} f_n = 1$. Let t be the greatest common divisor of those n for which $f_n > 0$. Let $\{ u_n \}$be another sequence such that $u_0 = 1$,

$u_n = \sum_{r=1}^{n} f_r u_{n-r}$ ($n \geq 1$) . Then $\underset{n \to \infty}{\text{Limit}}\, u_n = \mu$, where $\mu = \sum_{n=1}^{\infty} n f_r$.

Theorem

If a state j is recurrent , then as $n \to \infty$, $P_{jj}(nt) \to \frac{t}{\mu_{jj}}$, where j is periodic with t .

Proof

Let the state j be recurrent. Then $\mu_{jj} = \sum_n n f_{jj}(n)$is defined. Since $P_{jj}(n) = \sum_{r=1}^{n} f_{jj}(r) P_{jj}(n-r)$, we replace f_{jj} by f_n , $P_{jj}(n)$ by μ_n and μ_{jj} by μ in lemma 1.

Then we get $\underset{n\to\infty}{\text{Limit}}\, P_{jj}(nt) \to \frac{t}{\mu_{jj}}$.

Recurrent and Aperiodic State

If a state j is recurrent and aperiodic , then $P_{jj}(n) \to \frac{1}{\mu_{jj}}$ as $n\to\infty$ .The state is aperiodic , t = 1, then $P_{jj}(nt)$ becomes $P_{jj}(n) \to \frac{1}{\mu_{jj}}$ as $n\to\infty$.

Recurrent Null State

If a state j is recurrent null (whether periodic or aperiodic), then $P_{jj}(n) \to 0$ as $n\to\infty$. In the case of j as recurrent null , $\mu_{jj} = \infty$. So becomes $P_{jj}(n) \to 0$ as $n\to\infty$ .If state k is either transient or recurrent null , then for every j , $P_{jk}(n) \to 0$ as $n\to\infty$.

We know $P_{jk}(n) = \sum_{r=1}^{n} f_{jk}(r) P_{kk}(n-r)$.

Let n > m , then $P_{jk}(n) = \sum_{r=1}^{m} f_{jk}(r) P_{kk}(n-r) + \sum_{r=m+1}^{m} f_{jk}(r) P_{kk}(n-r)$

$$\leq \sum_{r=1}^{m} f_{jk}(r) P_{kk}(n-r) + \sum_{r=m+1}^{n} f_{jk}(r)$$

We know $\sum_{m=1}^{\infty} f_{jk}(m) < \infty$ and $\sum_{r=m+1}^{n} f_{jk}(r) \to 0$ and $P_{kk}(n-r) \to 0$ as $n\to\infty$, We get $P_{jk}(n) \to 0$ as $n\to\infty$.

Recurrent Non-null State

If state k is aperiodic , recurrent non-null , then $P_{jk}(n) \to \frac{F_{jk}}{\mu_{kk}}$ as $n\to\infty$. From Theorem , we have $P_{jk}(n) - \sum_{r=1}^{m} f_{jk}(r) P_{kk}(n-r) \leq \sum_{r=m+1}^{n} f_{jk}(r)$, Since j is aperiodic recurrent and non-null , then, $P_{kk}(n-r) \to \frac{1}{\mu_{kk}}$ as $n\to\infty$. Then , we get as $n , m\to\infty$,

$P_{jk}(n) - \frac{\sum_{r=1}^{m} f_{jk}(r)}{\mu_{kk}} \to 0$. Therefore , $P_{jk}(n) \to \frac{F_{jk}}{\mu_{kk}}$ as $n\to\infty$.

Applications of Markov Chain

The Gambler's Ruin problem

Consider a gambler who at each play of the game has probability P of winning one unit and probability q = 1-P of losing one unit. Assuming that successive plays of the game are independent, what is the probability that, starting with i units, the Gambler's fortune will reach N before reaching 0? (Ross, 1972).

If we let X_n denote the players fortune at time n , then the process $(X_n , n = 0,1,2,\ldots \}$ is a Markov Chain with transition probabilities $P_{00} = P_{NN} = 1$, $P_{i,i+1} = P = 1- P_{i,i-1}$, i = 1, 2, 3,…,N-1 .

This Markov Chain has three classes , namely , {0}, { 1,2,3,…,N-1 }, and {N}; the first and third class being recurrent and the second transient . Since each transient state is visited only finitely often, it follows that, after some finite amount of time, the Gambler will either attain his goal of N or go broke.

Let P_i, i = 0, 1, 2, …, N , denote the probability that , starting with i, the Gambler's fortune will eventually reach N . By conditioning on the outcome of the initial play of the game we obtain

$$P_i = pP_{i+1} + q\,P_{i-1}\,, \quad i = 1 ,2 , \ldots , N-1 .$$

or equivalently , since p + q = 1, $\quad p\,P_i + q\,P_i = pP_{i+1} + q\,P_{i-1}$,

or $$P_{i+1} - P_i = \frac{q}{p}\,(P_i - P_{i-1})\,, \quad i = 1 ,2 , \ldots , N-1 .$$

Hence , We obtain from the preceding line that , since $P_0 = 0$,

$$P_2 - P_1 = \frac{q}{p}\,(P_1 - P_0) = \left(\frac{q}{p}\right) P_1 ,$$

$$P_3 - P_2 = \frac{q}{p}\,(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1 ,$$

$$\vdots$$

$$P_i - P_{i-1} = \frac{q}{p}\,(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1,$$

$$\vdots$$

$$P_N - P_{N-1} = \frac{q}{p}\,(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1.$$

Adding the first i-1 of the equation yields

$$P_i - P_{i-1} = P_1 \left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^{i-1} \right]$$

or

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)} P_1, \text{if } \dfrac{q}{p} \neq 1 \\[2ex] iP_1, \text{if } \dfrac{q}{p} = 1 \end{cases}$$

Now, using the fact that $P_N = 1$, we obtain that

$$P_1 = \begin{cases} \dfrac{1-\left(\dfrac{q}{p}\right)}{1-\left(\dfrac{q}{p}\right)^N} P_1 & \text{, if } p \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{N} & \text{, if } p = \dfrac{1}{2} \end{cases}$$

And hence

$$P_1 = \begin{cases} \dfrac{1-\left(\dfrac{q}{p}\right)^i}{1-\left(\dfrac{q}{p}\right)^N} P_1 & \text{, if } p \neq \dfrac{1}{2} \\[2ex] \dfrac{1}{N} & \text{, if } p = \dfrac{1}{2} \end{cases} \qquad \ldots\ldots \quad (1.2)$$

Note that , as $N \to \infty$,

$$P_1 \to \begin{cases} 1-\left(\dfrac{q}{p}\right)^i & \text{,if } p > \dfrac{1}{2} \\[2ex] 0 & \text{, if } p \leq \dfrac{1}{2} \end{cases}$$

Thus, if $p > \frac{1}{2}$, there is a positive probability that the gambler's fortune will increase indefinitely; if $p \leq \frac{1}{2}$, the gambler will , with probability 1, go broke against an infinitely rich adversary.

Example

Suppose Mg Ba and Mg Hla decide to flip pennies the one coming closet to the wall wins. Mg Hla , being the better player , has a probability 0.6 of winning on each flip , If Mg Hla starts with five pennies and Mg Ba with ten ,then what is the probability that Mg Hla will wipe Mg Ba out ? What is the probability if Mg Hla starts with ten and Mg Ba with 20?

Solution

(a) The desired probability is obtained from equation (2.1) by letting $i = 5$, $N = 15$ and $p = 0.6$.

Hence, the desired probability is $P = \dfrac{1-\left(\frac{2}{3}\right)^{5}}{1-\left(\frac{2}{3}\right)^{15}} \approx 0.87$.

(b) The desired probability is obtained from equation (2.1) by letting i = 10 , N = 30 and p = 0.6 .

Hence, The desired probability is $P = \dfrac{1-\left(\frac{2}{3}\right)^{10}}{1-\left(\frac{2}{3}\right)^{30}} \approx 0.98$.

Drug Testing

For an application of the gambler's ruin problem to drug testing, suppose that two new drugs have been developed for treating a certain disease. Drug i has a cure rate P_i , i = 1 , 2 , in the sense that each patient treated with drug i will be cured with probability P_i . These cure rates are , however , not known and suppose we are interested in a method for deciding whether $P_1 > P_2$ or $P_2 > P_1$.

To decide upon one of these alternatives, consider the following test:

Pairs of patients are treated sequentially with one member of the pair receiving drug 1 and determined, and the testing stops when the cumulative number of cures using one of the drugs exceeds the cumulative number of cures when using the other by some fixed predetermined number.

More formally, let

$$X_j = \begin{cases} 1, & \text{if the patients in the } j^{th} \text{ pair to receive drug number 1 is cured .} \\ 0, & \text{otherwise} \end{cases}$$

$$Y_j = \begin{cases} 1, & \text{if the patients in the } j^{th} \text{ pair to receive drug number 2 is cured.} \\ 0, & \text{otherwise.} \end{cases}$$

For a predetermined positive integer M test stops after pair N where N is the first value of n such that either

$$X_1 + \cdots + X_n - (Y_1 + \cdots + Y_n) = M$$

or

$$X_1 + \cdots + X_n - (Y_1 + \cdots + Y_n) = -M .$$

In the former case, we then assert that $P_1 > P_2$ and in the latter that $P2 > P_1$.

In order to help ascertain whether the preceding is a good test , one thing we would like to know is the probability of it leading to an incorrect decision . That is for given P_1 and P_2 where $P1 > P_2$, what is the probability that the test will in correctly assert that $P_2 > P_1$?

To determine this probability, note that after each pair is checked the cumulative difference of curves using drug 1 versus drug 2 will either go up by 1 with probability

$P_1(1-P_2)$, i.e , the probability that drug 1 leads a cure and drug 2 does not or go down by 1 with probability $(1-P_1)P_2$, or remain the same with probability $P_1P_2 + (1-P_1) (1-P_2)$. Hence, if we only consider those pairs in which the cumulative difference changes, then the difference will go up by 1 with probability

$$P = P \{ \text{up } 1 / \text{up } 1 \text{ or down } 1 \} = \frac{P_1(1-P_2)}{P_1(1-P_2)+(1-P_1)P_2}$$

and down 1 with probability

$$q = 1 - p = \frac{P_2(1-P_1)}{P_1(1-P_2)+(1-P_1)P_2} .$$

Hence, the probability that the test will assert that $P_2 > P_1$ is equal to the probability that a gambler who wins each (one unit) bet with probability P will go down M before

going up M before going up M . But i = M , N = 2M , the probability is given by

$$P \{ \text{test asserts that } P_2 > P_1 \} = 1 - \frac{1-\left(\frac{q}{p}\right)^M}{1-\left(\frac{q}{p}\right)^{2M}}$$

$$= \frac{1}{1+\left(\frac{p}{q}\right)^M}$$

Thus, for instance, if $P_1 = 0.6$ and $P_2 = 0.4$ then the probability of an incorrect decision is 0.017 when M = 5 and reduces to 0.0003 when M = 10.

Conclusion

We have studied the transition probability function of the Markov Chain .The Gambler’s Ruin problem and drug testing have been also studied . Gambler's Ruin problem is very useful in practical world such as lottery, chess and lucky draw, etc …

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