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The Evolution of a Single-Phase Mode for Nonlinear Dispersive Waves

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Abstract

A method is devised to study the evolution of a system of waves which locally have a multi periodic structure. The present theory generalizes the results for the development of a single periodic wave train subject to large scale variations.

Key words: Evolution, periodic wave train, variation

Introduction

In this paper, the author reproduced the problem of a single – phase mode considered by Luke (1966) and Whitham (1965 a, b, 1967 a, b). This problem was reexamined and put in a form conveniently for extension to modes having more than one phase.

The Problem of a Single-Phase Mode

The present study reproduced the two – timing method due to Luke (1966) and Whitham (1965 a, b, 1967 a, b), for a single – phase mode. This method resembles the procedure used by (Kuzmak, 1959) for ordinary differential equations that are nearly periodic, but fully nonlinear. **First,** the Hamilton's variational principle was considered.

$$\delta J = \delta \int \int L(u, u_t, u_x)\, dt dx = 0, \tag{1}$$

with the suitable Lagrangian density

$$L = (u_t)^2/2 - (u_x)^2/2 - V(u), \tag{2}$$

and the author obtained the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + V'(u) = 0. \tag{3}$$

Second, the author introduced the slow variables X and T by the relations

$$X = \in x, \qquad T = \in t, \tag{4}$$

and assumed an asymptotic expansion of the form

$$u(x, t) = f(\theta, X, T) + \in u^{(1)}(\theta, X, T) + \in^2 u^{(2)}(\theta, X, T) + \in^3 u^{(3)}(\theta, X, T) + O(\in^4), \tag{5}$$

where $\in$ is a small positive parameter. The wave number θ_x and frequency - θ_t are allowed to depend on the slow variables X and T so that

$$-\theta_t = \omega(X, T), \qquad \theta x = \kappa(X, T), \tag{6}$$

together with the consistency condition

$$\kappa_T + \omega_X = 0, \tag{7}$$

which is equivalent to assuming that

$$\omega = -\theta t = -\Theta_T, \kappa = \theta_x = \theta_X \tag{8}$$

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where the phase variable

$$\theta = \in^{-1} \Theta\,(X, T). \tag{9}$$

Third, in terms of the independent variables θ, X and T, the author put the Klein – Gordon equation (3) in the expanded form

$$\omega^2 \frac{\partial^2}{\partial\theta^2}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\} - 2 \in \omega \frac{\partial^2}{\partial\theta\partial T}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^{4)})\}$$

$$- \in \frac{\partial\omega}{\partial T}\frac{\partial}{\partial\theta}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\} + \in^2 \frac{\partial^2}{\partial T^2}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\} -$$

$$-\kappa^2 \frac{\partial^2}{\partial\theta^2}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\} - 2 \in \kappa \frac{\partial^2}{\partial\theta\partial X}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\}$$

$$- \in \frac{\partial\kappa}{\partial X}\frac{\partial}{\partial\theta}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\} - \in^2 \frac{\partial^2}{\partial X^2}\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\}$$

$$- V'(f) + V''(f)[\in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)\} + V'''(f)\{f + \in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)]\, {}^{2}\!/_{2}$$

$$+ V''''\,(f)\,[\in u^{(1)} + \in^2 u^{(2)} + \in^3 u^{(3)} + O(\in^4)]\, {}^{3}\!/_{6} + \ldots = 0, \tag{10}$$

by using the assumed asymptotic expansion (5): From the expansion (10), for the leading-order, the author got the second-order homogeneous differential equation for the function f,

$$(\omega^2 - \kappa^2)\, f_{\theta 0} + V'\,(f) = 0. \tag{11}$$

From the expansion (10), for the first-order, the author obtained the second-order nonhomogeneous differential equation for the function $u^{(1)}$:

$$(\omega^2 - \kappa^2)\; u^{(1)}_{\theta 0} + V''\,(f)\, u^{(1)} = F_1, \tag{12}$$

where F_1 is defined by

$$F_1 = 2\omega\, f_{\theta T} + 2\kappa\, f_{\theta X} + w_T\, f_\theta + \kappa_X\, f_\theta. \tag{13}$$

Fourth, the author applied the two-timing method by treating f, $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, … as functions of the fast variable θ. and the slow variables X and T. From equation (11), then the author obtained the implicit solution

$$\theta = \sqrt{\omega^2 - \kappa^2} \int \frac{df}{\sqrt{2(E - V(f))}} - B, \tag{14}$$

where B = B (X, T), E = E (X, T) are the constants of integration. If f is periodic in θ with a constant period $2\mathcal{P}$, then the normalization condition is

$$\sqrt{\frac{\omega^2 - \kappa^2}{2}} \int \frac{df}{\sqrt{E - V(f)}} = 2\mathcal{P}\,, \tag{15}$$

which is a (dispersion) relation that holds between the frequency ω (X, T), wave number κ (X, T) and E (X, T). **Fifth**, the author considered the linear case, by letting V (u) = $u^2/2$ in (3). The periodic solution is $f = \sqrt{2E}\ \cos\theta$ with period $2\mathcal{P} = 2\pi$. Since the zeros of E – V (f) are $-\sqrt{2E}$ and $\sqrt{2E}$, the linear dispersion relation is

$$\omega^2 - \kappa^2 = 1, \tag{16}$$

which is independent of E and the amplitude is $\alpha = \sqrt{2E}$. **Sixth**, the author obtained the condition

$$\int_0^{2\mathcal{P}} F_1 f_\theta d\theta = \theta, \tag{17}$$

which will avoid the secular terms proportional to θ from $u^{(1)}$, $u_\theta^{(1)}$. **Seventh**, by using expression (13) in the secular condition (17), the author obtained the amplitude equation

$$\frac{\partial}{\partial T}\left\{\omega\sqrt{\frac{2}{\omega^2-\kappa^2}}\int\sqrt{E-V(f)}df\right\}+\frac{\partial}{\partial X}\left\{\kappa\sqrt{\frac{2}{\omega^2-\kappa^2}}\int\sqrt{E-V(f)}df\right\}=0. \tag{18}$$

Then the consistency condition (7), the dispersion relation (15), and the amplitude equation (18) form a coupled set of equations for the function E (X, T), wave number κ (X, T) and frequency ω (X, T).

Reexamination of the Problem of a Single-Phase Mode

In this paper, the author put in a form convenient for extension to modes have more than one phase, in the following manner. **First,** the author reconsidered the one dimensional Klein-Gordon equation

$$u_u - u_{xx} + V'(u) = 0, \tag{19}$$

which provides a relatively simple model for nonlinear dispersive waves. A permanent wave

$$u = f(\theta) \qquad \theta = \kappa x - \omega t, \tag{20}$$

(so that $\kappa = \partial\theta / \partial x$ is the wave number and $\omega = -\partial\theta / \partial t$ the frequency), is the solution of the second – order nonlinear equation

$$(\omega^2 - \kappa^2) f_{\theta 0} + V'(f) = 0, \tag{21}$$

Equation (21) can be solved by elementary means and has solutions periodic in θ for reasonable forms of V' (f) and has a solution given implicitly by the relation

$$\theta = \sqrt{\frac{\omega^2-\kappa^2}{2}}\int\frac{df}{\sqrt{E-V(f)}}, \tag{22}$$

where E (x, t) is the constant of integration. If the function u is periodic in θ with a fixed constant period $2\mathcal{P}$. The author had the dispersion relation

$$\sqrt{\frac{\omega^2-\kappa^2}{2}}\int\frac{df}{\sqrt{E-V(f)}} = 2\mathcal{P}, \tag{23}$$

Between the frequency ω (x, t), wave number κ (x, t) and E (x, t). **Second,** the author reconsidered the linear case, $V(u) = u^2/2$, that is the linear equation

$$u_u = u_{xx} + u = 0 \tag{24}$$

By assuming a solution of the form

$$f = \alpha \cos(2\pi\theta/P), \tag{25}$$

periodic in θ with a fixed constant period P, and constant amplitude α, the author found that $\alpha = \sqrt{2E}$ and the linear dispersion relation

$$\omega^2 - \kappa^2 = P^2 / (2\pi)^2. \tag{26}$$

Permanent waves (25) are very special solutions of (19) and my aim is to extend this type of solution to a more general class of solutions. **Third**, the author assumed the solution of the nonlinear Klein-Gordon equation (19) to be periodic in θ, with a fixed constant period $2\mathcal{P}$, and have slow space and time variations. Accordingly the author wrote

$$\theta = \in^{-1}\Theta\,(X, T), \qquad X = \in x, \qquad T = \in t, \tag{27}$$

where $\in$ is a small positive parameter, and the wave number κ and the frequency ω are now slowly varying functions derivable from the phase function θ as

$$\kappa = \theta_x = \Theta_X, \qquad \omega = -\theta_t = -\Theta_T. \tag{28}$$

For the existence of a rapid phase θ, the author had the consistency condition

$$\kappa_T + \omega_X = 0. \tag{29}$$

In terms of the fast variable θ, and slow variables X, T, the Klein – Gordon equation (19) can be written in the form

$$m\, u_{\theta\theta} + V'(u) = \in[2(\omega u_{\theta T} + \kappa u_{\theta X}) + M\, u_\theta] + \in^2 (u_{XX} - u_{TT}), \tag{30}$$

where the functions m and M are denoted by the relations

$$m = \omega^2 - \kappa^2 = \Theta_T^2 - \Theta_X^2. \qquad M = \omega_T + \kappa_x = \Theta_{XX} - \Theta_{TT}. \tag{31}$$

Fourth, the author wrote the functions u, m and M in the formal power series expansions,

$$u = f + \in u^{(1)} + \in^2 u^{(2)} + \dots, \tag{32}$$

$$m = g + \in m^{(1)} + \in^2 m^{(2)} + \dots, \tag{33}$$

$$M = g + \in M^{(1)} + \in^2 M^{(2)} \dots, \tag{34}$$

and put the Klein-Gordon equation (19) in terms of the fast and slow variable θ, X, T as

$$\begin{aligned}
&(g + \in m^{(1)} + \in^2 m^{(2)} + \dots)(f + \in u^{(1)} + \in^2 u^{(2)} + \dots)_{\theta\theta} \\
&+ \{V'(f) + (\in u^{(1)} + \in^2 u^{(2)} + \dots) V''(f) + (\in u^{(1)} + \in^2 u^{(2)} + \dots)^2 V'''(f)/2 \\
&\qquad + (\in u^{(1)} + \in^2 u^{(2)} + \dots)^3 V'''(f)/6 + \dots\} \\
&= \in \{2/\omega\,(f + \in u^{(1)} + \in^2 u^{(2)} + \dots)_{\theta T} + \kappa\,(f + \in u^{(1)} + \in^2 u^{(2)} + \dots)_{\theta X}] \\
&\quad + (G + \in M^{(1)} + \in^2 M^{(2)} + \dots)(f + \in u^{(1)} + \in^2 u^{(2)} + \dots)_\theta \} \\
&\quad + \in^2 [(f + \in u^{(1)} + \in^2 u^{(2)} + \dots)_{XX} - (f + \in u^{(1)} + \in^2 u^{(2)} + \dots)_{TT}].
\end{aligned} \tag{35}$$

From the expansion (35), for the leading-order, the author got

$$g\, f_{\theta\theta} + V'(f) = 0, \tag{36}$$

which is a second-order, nonlinear, homogeneous differential equation for the unknown function f. From the expansion (35), for the first-order, the author got

$$Lu^{(1)} = F^{(1)}, \tag{37}$$

which is a second-order, linear, non-homogeneous differential equation for the unknown function $u^{(1)}$, where the non-homogeneous term

$$F^{(1)} = F^{(1)}\,(f;\, m^{(1)};\, G), \tag{38}$$

is given by the expression

$$F^{(1)} = -m^{(1)}\, f_{\theta\theta} + 2\,(\omega f_{\theta T} + \kappa f_{\theta X}) + G f_{\theta}, \tag{39}$$

and the self-adjoin operation L is given by the expression

$$L = g\,\frac{d^2}{d\theta^2} + V''\,(f) \tag{40}$$

From the expansion (35), for the second – order, the author got

$$Lu^{(2)} = F^{(2)} \tag{41}$$

which is a second-order, linear, non-homogeneous differential equation for the unknown function $u^{(2)}$, where the non-homogeneous term

$$F^{(2)} = F^{(2)}\,(f,\, u^{(1)}\,;\, m^{(1)}\,,\, m^{(2)}\,;\, G,\, M^{(1)}), \tag{42}$$

is given by the expression

$$F^{(2)} = -m^{(1)}\; u^{(1)}_{\theta 0} - m^{(2)}\, f_{\theta 0} + 2\left(\omega u^{(1)}_{\theta T} + \kappa u^{(1)}_{\theta X}\right) - \; {u^{(1)}}^{2}\;\; V'''\,(f)\,/\,2$$

$$+ G u^{(1)}_{\theta} + M^{(1)} f_{\theta} + f_{xx} - f_{TT}\,. \tag{43}$$

From the expansion (35), for the third – order, the author got

$$Lu^{\,(3)} = F^{(3)} \tag{44}$$

which is a second-order, linear, non-homogeneous differential equation for the unknown function $u^{(3)}$, where the non-homogeneous term

$$F^{(3)} = F^{(3)}\,(f,\, u^{(1)},\, u^{(2)};\, m^{(1)},\, m^{(2)},\, m^{(3)};\, G,\, M^{(1)},\, M^{(2)}), \tag{45}$$

is given by the expression

$$F^{(3)} = -\, m^{(1)}\; u^{(2)}_{\theta 0} - m^{(2)}\; u^{(1)}_{\theta 0} - m^{(3)}\, f_{\theta 0} - u^{(1)}\, u^{(2)}\, V'''\,(f) - \, {u^{(1)}}^{3}\, V''''\,(f)\,/\,6$$

$$+\, 2\left(\omega u^{(2)}_{\theta T} + \kappa u^{(2)}_{\theta X}\right) + G\, u^{(2)}_{0} + M^{(1)}\, u^{(1)}_{0} + M^{(2)}\, f_0 + u^{(1)}_{XX} - u^{(1)}_{TT}\,. \tag{46}$$

In general, from the expansion (35), for the n^{th} order, $(n \geq 1)$, the author had

$$Lu^{\,(n)} = F^{(n)}, \tag{47}$$

which is a second-order. Linear, non-homogeneous equation for the unknown function $u^{(n)}$, and the non-homogeneous term $F^{(n)}$ is given by

$$F^{(n)} = F^{(n)}\,(f,\, u^{(1)},\, u^{(2)},\, \ldots,\, u^{(n-1)};\, m^{(1)},\, m^{(2)},\, m^{(3)},\, \ldots\,,\, m^{(n)};\, G,\, M^{(1)},\, M^{(2)},\, \ldots\,,\, M^{(n-1)}). \tag{48}$$

Fifth, the author obtained the leading – order solution f given implicitly by

$$\theta = \sqrt{\frac{g}{2}}\int \frac{df}{\sqrt{E - V(f)}}\,, \tag{49}$$

where E. the amplitude of the wave, is a function of the slow variables X, T. The fundamental dispersion relation $\omega^2 - \kappa^2 = g\,(E)$ is found by taking the function f which is periodic in θ with a fixed constant period $2\mathcal{P}$, so that

$$\sqrt{\frac{g}{2}}\int\frac{df}{\sqrt{E-V(f)}}=2\mathcal{P}\ . \tag{50}$$

Sixth, the author obtained two linearly independent solutions $\omega=\omega_1=f_0$ and $\omega=\omega_2=f_g+(\theta/(2g))\,\omega_1$ of the homogeneous differential equation

$$L\omega=0. \tag{51}$$

Equation (51) is the corresponding homogeneous differential equation of the non-homogeneous differential (37) or (41) or (44) or (47). **Seventh**, the author obtained the first relationship

$$g\left[\omega_1 u_\theta^{(n)}-\omega_{1\theta}u^{(n)}\right]_0^\theta=\int_0^\theta F^{(n)}\omega_1 d\theta, \qquad (n\ge 1). \tag{52}$$

and the second relationship

$$g\left[\omega_2 u_\theta^{(n)}-\omega_{2\theta}u^{(n)}\right]_0^\theta=\int_0^\theta F^{(n)}\omega_2 d\theta, \qquad (n\ge 1). \tag{53}$$

to obtain explicit solution $u^{(n)}$. **Eight,** the author obtained the condition

$$\int_0^{2\mathcal{P}} F^{(n)}\omega_1 d\theta=0, \qquad n\ge 1, \tag{54}$$

which is the condition for the existence of a periodic function $u^{(n)}$. **Ninth**, the author obtained the condition

$$\int_0^{2\mathcal{P}} F^{(n)}\omega_2 d\theta=-\mathcal{P}\omega_{1\theta}(0)u^{(n)}(0). \qquad n\ge 1, \tag{55}$$

which is appropriate to ensure periodicity for the function $u^{(n)}$. **Tenth**, by using the normalization condition

$$u^{(n)}(0)=0=u_\theta^{(n)}(0), \tag{56}$$

The author found that the unique periodic solution is

$$u^{(n)}=\frac{1}{gW}\left[\omega_2\int_0^\theta f^{(n)}\omega_1 d\theta-\omega_1\int_0^\theta f^{(n)}\omega_2 d\theta\right], \qquad (n\ge 1), \tag{57}$$

where the function $W = W(\omega_1, \omega_2)$ is the constant Wronskian of ω_1 and ω_2, defined by

$$W=\omega_1\,\omega_{2\theta}-\omega_2\,\omega_{1\theta}\ . \tag{58}$$

Moreover, the two secular conditions the author obtained the condition

$$\int_0^{2\mathcal{P}} F^{(n)} f_\theta d\theta=0, \tag{59}$$

$$\int_0^{2\mathcal{P}}\left\{F^{(n)}f_g-\frac{1}{2g}\int_0^\theta F^{(n)}f_{\theta'}d\theta'\right\}d\theta=0. \tag{60}$$

Eleventh, the author considered the $u^{(1)}$ problem, and it is found that the two secular conditions (59) and (60) reduce to

$$m^{(1)}=0, \tag{61}$$

$$G\int_0^{2\mathcal{P}} f_\theta^2 d\theta+\frac{\partial}{\partial T}\left(\int_0^{2\mathcal{P}}\omega f_\theta^2 d\theta\right)+\frac{\partial}{\partial X}\left(\int_0^{2\mathcal{P}}\kappa f_\theta^2 d\theta\right)=0\,. \tag{62}$$

Twelfth, the author considered the $u^{(2)}$ problem, and it is found that the two secular conditions (59) and (60) reduce to

$$M^{(1)} = 0: \tag{63}$$

$$-m^{(2)} \int_0^{2\mathcal{P}} f_{\theta 0} \left\{ f_g + \frac{\theta}{2g} f_\theta \right\} d\theta$$

$$= \int_0^{2\mathcal{P}} \left\{ -Gu_\theta^{(1)} - 2(\omega u_{\theta T}^{(1)} + \kappa u_{\theta X}^{(1)}) + \frac{1}{2} u^{(1)^2} V'''(f) + f_{TT} - f_{XX} \right\} \left\{ f_g + \frac{\theta}{2g} f_\theta \right\} d\theta. \tag{64}$$

In general, the author found that $m^{(1)} = M^{(1)} = 0$ when n is an odd positive integer. Moreover, $u^{(n)}$ is an odd function of θ when n is an odd positive integer, and $u^{(n)}$ is an even function of θ when n is an even positive integer. **Thirteenth**, the author reconsidered the linear case (24), and it is found that

$$g = \mathcal{P}^2 / \pi^2 = \text{constant}, \qquad G = -(\omega E_T + \kappa E_X) / E. \tag{65}$$

Conclusion

In this paper, a method has been developed for investigating the evolution of nonlinear waves which have many local periodicities. The detailed analysis was applied to only one nonlinear wave problem, the one dimensional Klein-Gordon equation. I hope that it will be possible to construct the averaged Lagrangian by appealing to the basic nonlinear problem as a special case, for example, weak interactions and wave propagation in slowly varying medium.

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