Finite Difference Method for Elliptic Partial Differential Equation

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Abstract

This paper is considered on the Dirichlet boundary value problems for a two-dimensional elliptic equation. The existence and uniqueness of the solutions of the Dirichlet boundary value problems are discussed. Different approximations are used to derive a finite difference formula. The consistency and the stability of the finite difference formula are investigated.

Keywords: elliptic equation, finite difference, Dirichlet boundary value problem

Introduction

The resulting finite difference numerical method for solving partial differential equations (Evans, L. C. 1998) have extremely broad applicability and can, with proper care, be adapted to most problems that arise in mathematics and its many applications.

Existence of Weak Solution of Elliptic Problem

We consider the Dirichlet boundary value problem (Evans, L. C. 1998) for poisson equation in two space dimensions

$$-\Delta u = f \text{ in } \Omega \subset R^2 \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega \tag{1.2}$$

Multiplying the elliptic equation (1.1) by a test function $v \in H_0^1(\Omega)$ and integrating over Ω, we have

$$-\int_\Omega v\Delta u\, dx = \int_\Omega fv\, dx \tag{1.3}$$

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega fv\, dx. \tag{1.4}$$

Then, we define a bilinear from *B* by

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx \tag{1.5}$$

and a linear function *F* by

$$F(v) = \int_\Omega fv\, dx. \tag{1.6}$$

Then we study the continuity and the coercivity of *B*.

Continuity of B

$$|B(u,v)| = |\int_\Omega \nabla u \cdot \nabla v\, dx|$$

By using Hölder's inequality (Thomas, J. W. 1998), we have

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$$
\begin{aligned}
|B(u,v)| &\le \|\nabla u\|_2 \|\nabla v\|_2 \\
&\le \left(\|\nabla u\|_2^2\right)^{\frac{1}{2}} \left(\|\nabla v\|_2^2\right)^{\frac{1}{2}} \\
&\le \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}} \left(\|v\|_2^2 + \|\nabla v\|_2^2\right)^{\frac{1}{2}} \\
&= \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \qquad \text{for all } u,v \in H_0^1(\Omega).
\end{aligned}
$$

Therefore $B(u, v)$ is continuous.

Coercivity of B

$$
\begin{aligned}
B(u,u) &= \int_\Omega |\nabla u|^2 \, dx \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx \\
&= \frac{1}{2}\left(\|\nabla u\|_2^2\right) + \frac{1}{2}\left(\|\nabla u\|_2^2\right).
\end{aligned}
$$

By using Poincaré's inequality(Thomas, J. W. 1998), we have

$$
\begin{aligned}
B(u,u) &\ge \frac{1}{2C_1}\|u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\
&\ge \min\left\{\frac{1}{2C_1}, \frac{1}{2}\right\}\left(\|u\|_2^2 + \|\nabla u\|_2^2\right) \\
&= C_2 \|u\|_{H_0^1(\Omega)}^2, \qquad \text{for all } u \in H_0^1(\Omega),
\end{aligned}
$$

where $C_2 = \min\left\{\frac{1}{2C_1}, \frac{1}{2}\right\}$.

Therefore $B(u, u)$ is coercive.

By Lax-Milgram Theorem(Thomas, J. W. 1998), (1.1) has a unique solution.

A Priori Bound

Multiplying (1.1) by a test function $u - u_D \in H_0^1(\Omega)$ and integrating over Ω, we have

$$
-\int_\Omega \Delta u(u - u_D)dx = -\int_\Omega f(u - u_D)dx
$$

$$
\begin{aligned}
\int_\Omega |\nabla u|^2 \, dx &= \int_\Omega \nabla u \cdot \nabla u_D dx + \int_\Omega f\, u\, dx - \int_\Omega f\, u_D dx \\
&\le \int_\Omega |\nabla u||\nabla u_D| \, dx + \int_\Omega |f||u| \, dx \\
&\quad - \int_\Omega |f||u_D| \, dx.
\end{aligned}
$$

By using Young's inequality(Thomas, J. W. 1998), we have

$$\|\nabla u\|_2^2 \le \frac{\varepsilon_1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2\varepsilon_1}\int_\Omega |\nabla u_D|^2\,dx + \frac{\varepsilon_2}{2}\int_\Omega |u|^2\,dx$$

$$+\frac{1}{2\varepsilon_2}\int_\Omega |f|^2\,dx + \frac{\varepsilon_3}{2}\int_\Omega |u_D|^2\,dx + \frac{1}{2\varepsilon_3}\int_\Omega |f|^2\,dx$$

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \le \frac{\varepsilon_1}{2}\|\nabla u\|_2^2 + \frac{\varepsilon_2}{2}\|u\|_2^2 + \frac{1}{2\varepsilon_1}\|\nabla u_D\|_2^2 + \frac{\varepsilon_3}{2}\|u_D\|_2^2$$

$$+\left(\frac{1}{2\varepsilon_2} + \frac{1}{2\varepsilon_3}\right)\|f\|_2^2 .$$

By using Poincaré inequality(Thomas, J. W. 1998), we have

$$\frac{1}{2C_3}\|u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \le \frac{\varepsilon_1}{2}\|\nabla u\|_2^2 + \frac{\varepsilon_2}{2}\|u\|_2^2 + \frac{1}{2\varepsilon_1}\|\nabla u_D\|_2^2$$

$$+\frac{\varepsilon_3}{2}\|u_D\|_2^2 + \left(\frac{1}{2\varepsilon_2} + \frac{1}{2\varepsilon_3}\right)\|f\|_2^2 .$$

For sufficiently small ε_1, ε_2, ε_3,

$$\left(\frac{1}{2C_3} - \frac{\varepsilon_2}{2}\right)\|u\|_2^2 + \left(\frac{1}{2} - \frac{\varepsilon_1}{2}\right)\|\nabla u\|_2^2 \le \frac{1}{2\varepsilon_1}\|\nabla u_D\|_2^2 + \frac{\varepsilon_3}{2}\|u_D\|_2^2 + \left(\frac{1}{2\varepsilon_2} + \frac{1}{2\varepsilon_3}\right)\|f\|_2^2$$

$$C_4\left(\|u\|_2^2 + \|\nabla u\|_2^2\right) \le C_5\left(\|u_D\|_2^2 + \|\nabla u_D\|_2^2\right) + C_6\|f\|_2^2$$

where

$$C_4 = \min\left(\frac{1}{2C_3} - \frac{\varepsilon_2}{2}, \frac{1}{2} - \frac{\varepsilon_1}{2}\right)$$

$$C_5 = \max\left(\frac{1}{2\varepsilon_1}, \frac{\varepsilon_3}{2}\right)$$

$$C_6 = \min\left(\frac{1}{2\varepsilon_2}, \frac{1}{2\varepsilon_3}\right)$$

$$\|u\|_{H_0^1}^2 \le \max\left(C_5, C_6\right)\left(\|u_D\|_{H_0^1}^2 + \|f\|_{H_0^1}^2\right)$$

$$\|u\|_{H_0^1} \le C_7\left(\|u_D\|_{H_0^1} + \|f\|_{H_0^1}\right)$$

where

$$C_7 = \sqrt{\frac{\max\left(C_5, C_6\right)}{C_4}} .$$

Finite Difference Method

We consider the equation

$$-\Delta u = f(x),\ in\ \Omega = (0,1)^2 \qquad (1.7)$$

with the boundary conditions

$$u = 0 \ \text{ on } \ \partial\Omega. \qquad (1.8)$$

Let the grid spacing in x and y be $h = \frac{1}{N}, k = \frac{1}{M}$ and the set of inner grid points is denoted by

$$\partial\Omega_{h,k} = \{(x, y) \mid x = 0, y = jk, j = 0, \ldots, M\}$$
$$\cup \{(x, y) \mid x = 1, y = jk, j = 0, \ldots, M\}$$
$$\cup \{(x, y) \mid x = ih, y = 0, i = 0, \ldots, N\}$$
$$\cup \{(x, y) \mid x = ih, y = 1, i = 0, \ldots, N\}.$$

Using the following difference approximations

$$\left(D_x^+ D_x^-\right)u = \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2} + O(h^2)$$

$$\left(D_y^+ D_y^-\right)u = \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{k^2} + O(k^2)$$

in (1.7), we have

$$-\Delta u(x_i, y_j) = f_{ij}$$

$$-\left(D_x^+ D_x^-\right)u\left(x_i, y_j\right) - \left(D_y^+ D_y^-\right)u\left(x_i, y_j\right) = f_{ij}$$

$$\frac{-u_{i-1,j} + 2u_{ij} - u_{i+1,j}}{h^2} + \frac{-u_{i,j-1} + 2u_{ij} - u_{i,j+1}}{k^2} = f_{ij} \qquad (1.9)$$

$$u_{0j} = u_{Nj} = u_{i0} = u_{iM} = 0. \qquad (1.10)$$

5-Point Method

If we choose $h = k$, then we have the system of linear equations become

$$-\Delta u\left(x_i, y_j\right) = \frac{-u_{i-1,j} - u_{i,j-1} + 4u_{ij} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{ij}. \qquad (1.11)$$

The system of linear equations (1.11) can be written in the matrix form

$$AU = F$$

where

$$A=\frac{1}{h^2}\begin{pmatrix} S & -I & & & \\ -I & S & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & S & -I \\ & & & -I & S \end{pmatrix},\quad S=\begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}$$

$$U = (u_{11}, u_{21}, \ldots, u_{N-1,1}, u_{12}, \ldots, u_{N-1,2}, \ldots, u_{N-1,N-1})^T$$

$$F = (f_{11}, f_{21}, \ldots, f_{N-1,1}, \ldots, f_{N-1,N-1})^T.$$

Consistency

The local truncation error of the finite difference method (1.11) is given by

$$\varepsilon_{i,j} = \frac{1}{h^2}\left[-u_{i-1,j}-u_{i,j-1}+4u_{i,j}-u_{i+1,j}-u_{i,j+1}\right]$$

$$=\frac{1}{h^2}\Bigg[-\left(u_{i,j}-hu_x(x_i,y_j)+\frac{h^2}{2}u_{xx}(x_i,y_j)-\frac{h^3}{6}u_{xx}(x_i,y_j)+\cdots\right)$$

$$-\left(u_{i,j}-hu_y(x_i,y_j)+\frac{h^2}{2}u_{yy}(x_i,y_j)-\frac{h^3}{6}u_{yyy}(x_i,y_j)+\cdots\right)+4u_{i,j}$$

$$-\left(u_{i,j}+hu_x(x_i,y_j)+\frac{h^2}{2}u_{xx}(x_i,y_j)+\frac{h^3}{6}u_{xx}(x_i,y_j)+\cdots\right)$$

$$-\left(u_{i,j}+hu_y(x_i,y_j)+\frac{h^2}{2}u_{yy}(x_i,y_j)+\frac{h^3}{6}u_{yyy}(x_i,y_j)+\cdots\right)\Bigg]$$

$$=\frac{1}{h^2}\Bigg[-\frac{2h^2}{2}u_{xx}(x_i,y_j)-\frac{2h^4}{24}u_{xxxx}(x_i,y_j)+\cdots$$

$$-\frac{2h^2}{2}u_{yy}(x_i,y_j)-\frac{2h^4}{24}u_{yyyy}(x_i,y_j)+\cdots\Bigg]$$

$$\varepsilon_{i,j}=-\frac{h^2}{12}u_{xxxx}(x_i,y_j)-\frac{h^2}{12}u_{yyyy}(x_i,y_j)+O(h^4).$$

Therefore, the finite difference method (1.11) is consistent of order 2.

Stability.

Let λ be the eigenvalue (Evans, L. C. 1998) of A and u be the corresponding eigenvector to λ.

$$Au-\lambda u = 0$$

$$-D_x^+D_x^-u-D_y^+D_y^-u-\lambda u = 0.$$

Multiplying the equation by u and summing, we obtain

$$-\sum\left(D_x^+D_x^-u(x,y)\right)u(x,y)-\sum\left(D_y^+D_y^-u(x,y)\right)u(x,y)-\lambda\sum\left(u(x,y)\right)^2=0$$

$$\sum\left(\left(D_x^-u\right)(x,y)\right)^2+\sum\left(\left(D_y^-u\right)(x,y)\right)^2-\lambda\sum\left(u(x,y)\right)^2=0 \qquad (1.12)$$

But

$$u(x,y)=h\sum_{z\le x}\left(D_x^-u\right)(z,y)$$

$$\sum\left(u(x,y)\right)^2=h^2\sum\left|\sum_{z\le x}\left(D_x^-u\right)(z,y)\right|^2 .$$

Using Cauchy-Schwarz inequality, we have

$$\sum\left(u(x,y)\right)^2\le\sum\left|\left(D_x^-u\right)(x,y)\right|^2. \qquad (1.13)$$

Similarly, we have

$$\sum\left(u(x,y)\right)^2\le\sum\left|\left(D_y^-u\right)(x,y)\right|^2. \qquad (1.14)$$

Multiplying (1.12) by *h* and using (1.13), (1.14), we have

$$2\,\|u\|_{2,h}^2-\lambda\,\|u\|_{2,h}^2\le 0$$

and

$$\begin{aligned} 2 &\le \lambda \\ \|A^{-1}\|_2 &= \rho\,(A^{-1}) \\ &= (\lambda_{\min})^{-1} \\ &\le \frac{1}{2}. \end{aligned}$$

Therefore, the finite difference method (1.11) is stable in 2-norm.

Conclusion

It is then necessary to resort to numerical or approximation methods in order to deal with the problems that cannot be solved analytically. In view of the wide-spread accessibility of today’s high speed electronic computers, numerical and approximation methods are becoming increasingly important and useful in applications.

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