Abstract Cones by Means of Monoid

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Abstract

In this paper, abstract cones from a unital semi-group are constructed and their properties are investigated. This research work is a study of abstract cone on the commutative monoid. There are two parts in this research paper. The first part concerns preliminaries on the fundamental concepts of the monoid and then we construct the cone on the commutative monoid.

Keywords: semi-group, commutative, monoid, abstract cone.

Introduction

There are many mathematicians who studied cone and related properties in various kinds of mathematics. Among them, cone is one of the fields which can be applied in many areas of applications. Abstract cone on commutative monoid is a kind of research for mathematicians. They are T. Ando (1962), F. Khojasteh (2013), Y. Sang (2013), and W. W. Subramanian (2012) extend the abstract cone on semi-group. This research work is a study of abstract cone on the commutative monoid. There are two parts in this research paper. The first part concerns preliminaries on the fundamental concepts of the monoid and then we construct the cone on the commutative monoid. Some usual sets to be investigated whether they are cones or not.

Preliminaries

Definition. If C is a nonempty set, then a ***binary operation*** on C is a function from $C \times C$ into C. If the binary operation is denoted by $*$, then we use the notation, if $(a, b) \in C \times C$ is mapped to C under the binary operation $*$.

The usual addition + is a binary operation on the set of real numbers $\mathbb{R}$. The usual multiplication $\cdot$ is a different binary operation on $\mathbb{R}$. In this example, we could replace by any of the sets $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{R}^+$ or $\mathbb{Z}^+$.

Let $\mathrm{M}(\mathbb{R})$ be the set of all matrices with real entries. The usual matrix addition + is not a binary operation on this set since $\mathrm{A}+\mathrm{B}$ is not defined for any ordered pair (A, B) of matrices having different numbers of rows or of columns.

Definitions. A binary operation $*$ is called ***commutative*** if $a*b = b*a$ for all a, b and a binary operation $*$ is called ***associative*** if $a*(b*c) = a*(b*c)$ for all a, b, c.

Addition on natural numbers is both commutative and associative binary operation. If we use $*$ on $\mathbb{N}$ by $a*b = a^2 + b$ then it is easy to see that is a binary operation on $\mathbb{N}$ but is not associative.

A nonempty set C together with a binary operation $*$ is called a ***semi-group*** if $a*(b*c) = a*(b*c)$ for all $a, b, c \in C$.

The set $\omega = \{0, 1, 2, ...\}$ of all natural numbers including 0 forms a semi-group $(\omega, +)$, under the usual addition of natural numbers and the semi-group $(\omega, .)$, under the usual multiplication of natural numbers.

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The set $\mathbb{N} = \{0, 1, 2, ...\}$ of all positive natural numbers excluding 0 gives the semi-group $(\mathbb{N}, +)$, under the usual addition and the semi-group $(\mathbb{N}, .)$, under the usual multiplication.

Definition. Let C be a nonempty set. There are two extremely simple semi-group structures on C with the multiplication given by $x * y = x$ for $x, y \in C$, $(C, *)$ is called the ***left zero semi-group*** over S.

Let C be any set and denote by $A_f(C)$ the set of all finite nonempty subsets of C. Clearly, $A_f(C)$ is a semi-group under the operation of taking the union of two sets.

Let $C = \{0, 1, 2\}$ and define on $*C$ by $a * b = |a - b|$. Then the set C is not a semi-group under the operation $*$.

Indeed $1 * (1 * 2) = 1 * 1 = 0$

$(1 * 1) * 2 = 0 * 2 = 2$

and, hence $1 * (1 * 2) \neq (1 * 1) * 2$.

Proposition. A semi-group can have at most one identity.

Theorem. Cancellation laws may not hold in a semi-group.

Proof. Consider the set M of all 2×2 matrices over integers under matrix multiplication, which forms a semi-group.

If we take $\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathrm{B} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathrm{C} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$,

then clearly $\mathrm{AB} = \mathrm{AC} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But $\mathrm{B} \neq \mathrm{C}$.

Definition. A semi-group C is ***commutative*** or ***abelian*** if $x * y = y * x$ for all $x, y \in C$.

Let $C = \{(a, b) \mid a, b \text{ are rationals}, a \neq 0\}$, define an operation $*$ on C by $(a, b) * (c, d) = (ac, ad + b)$.

Then C is not commutative or abelian for

$(1, 2) * (3, 4) = (3, 4 + 2) = (3, 6)$,

and $(3, 4) * (1, 2) = (3, 6 + 4) = (3, 10)$.

Thus $(1, 2) * (3, 4) = (3, 4) * (1, 2)$.

Definition. A ***monoid*** is a semi-group with an identity element. That is, let C be semi-group under the operation $*$. For all $x \in C$ there exists $e \in C$ such that $x * e = e * x = x$.

Basic Concepts on Abstract Cones

Let $\mathbb{R}^+$ be the set of all real numbers. An abstract cone is analogous to a real vector space, except that we take the non-negative real numbers as scalars. Since the set of non-negative real numbers does not form a field, we have to replace the vector space law v +

$(-v)=0$ by a cancelation law $v+u=w+u \Rightarrow v=w$. We also require strictness, which means that no non-zero element has a negative.

Definition. An ***abstract cone*** is a set C together with two binary operation $+: C\times C \to C$ and $\cdot : \mathbb{R}^{+}\times C \to C$ and a distinguished element $0\in C,$ satisfying the following laws for all $v, w, u \in C$ and $\lambda, \mu \in \mathbb{R}^{+}$ where $\mathbb{R}^{+}=\{x \mid x\geq 0, x\in\mathbb{R}\}$ such that

(A_1) $0+v=v$

(A_2) $v+(w+u)=(v+w)+u$

(A_3) $v+w=w+v$

(A_4) $v+u=w+u \Rightarrow v=w$ (cancelation)

(A_5) $v+w=0 \Rightarrow v=w=0$ (strictness)

(A_6) $1v=v$ and $0v=0$

(A_7) $(\lambda+\mu)v=\lambda v+\mu v$

(A_8) $\lambda(v+w)=\lambda v+\lambda w.$

Example. Let $C=\{(x_1,\ldots,x_n) \mid x_1\geq 0,\ldots,x_n\geq 0\}\subset \mathbb{R}^n$. Then C is an abstract cone in $\mathbb{R}^n$.

For if $(0,\ldots,0)\in C, (x_1,\ldots,x_n)\in C.$ Then

$$\begin{aligned}(0,\ldots,0)+(x_1,\ldots,x_n) &= (0+x_1,\ldots,0+x_n)\\ &=(x_1,\ldots,x_n).\end{aligned}$$

Hence (A_1) holds.

Let $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n)$ and $(z_1,\ldots,z_n)\in C.$ Then

$$\begin{aligned}\big((x_1,\ldots,x_n)+(y_1,\ldots,y_n)\big)+(z_1,\ldots,z_n) &= (x_1+y_1,\ldots, x_n+y_n)+(z_1,\ldots,z_n)\\ &=(x_1+y_1+z_1,\ldots, x_n+y_n+z_n).\\ (x_1,\ldots,x_n)+\big((y_1,\ldots,y_n)+(z_1,\ldots,z_n)\big) &= (x_1,\ldots, x_n)+(y_1+z_1,\ldots,y_n+z_n)\\ &=(x_1+y_1+z_1,\ldots, x_n+y_n+z_n).\end{aligned}$$

Hence (A_2) holds.

Let $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)\in C$. Then

$$(x_1,\ldots,x_n)+(y_1,\ldots,y_n) = (x_1+y_1,\ldots,x_n+y_n) = (y_1+x_1,\ldots,y_n+x_n) = (y_1,\ldots,y_n)+(x_1,\ldots,x_n).$$

Hence (A_3) holds.

Let $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n)$ and $(z_1,\ldots,z_n)\in C.$ Then

$$(x_1,\ldots,x_n)+(y_1,\ldots,y_n)=(z_1,\ldots,z_n)+(y_1,\ldots,y_n)\Rightarrow (x_1,\ldots,x_n)=(z_1,\ldots,z_n).$$

Hence (A_4) holds.

Let $(x_1,...,x_n)$ and $(y_1,...,y_n) \in C.$ Then

$(x_1,...,x_n)+(y_1,...,y_n)=0 \Rightarrow (x_1,...,x_n)=(y_1,...,y_n)=0.$

Hence (A_5) holds.

Let $(x_1,...,x_n) \in C.$

$1(x_1,...,x_n)=(1x_1,...,1x_n)=(x_1,...,x_n)$ for all $1 \in \mathbb{R}^+.$

$0(x_1,...,x_n)=(0x_1,...,0x_n)=(0,...,0)$ for all $0 \in \mathbb{R}^+.$

Hence (A_6) holds.

Let $(x_1,...,x_n) \in C.$ Then

$$
\begin{aligned}
(\lambda+\mu)(x_1,...,x_n) &= \big((\lambda+\mu)x_1,...,(\lambda+\mu)x_n\big)\\
&= (\lambda x_1+\mu x_1,...,\lambda x_n+\mu x_n)\\
&= \lambda(x_1,...,x_n)+\mu(x_1,...,x_n) \text{ for all } \lambda,\mu \in \mathbb{R}^+.
\end{aligned}
$$

Hence (A_7) holds.

Let $(x_1,...,x_n)$ and $(y_1,...,y_n) \in C$.

$$
\begin{aligned}
\lambda\big((x_1,...,x_n)+(y_1,...,y_n)\big) &= \lambda(x_1+y_1,...,x_n+y_n)\\
&= \big(\lambda(x_1+y_1),...,\lambda(x_n+y_n)\big)\\
&= (\lambda x_1+\lambda y_1,...,\lambda x_n+\lambda y_n).
\end{aligned}
$$

Again $\lambda(x_1,...,x_n)+\lambda(y_1,...,y_n) = (\lambda x_1,...,\lambda x_n)+(\lambda y_1,...,\lambda y_n)$

$$= (\lambda x_1+\lambda y_1,...,\lambda x_n+\lambda y_n).$$

So $\lambda\big((x_1,...,x_n)+(y_1,...,y_n)\big) = \lambda(x_1,...,x_n)+\lambda(y_1,...,y_n)$ for all $\lambda \in \mathbb{R}^+$.

Hence (A_8) holds.

Therefore C is an abstract cone in $\mathbb{R}^n$.

Abstract Cones of Usual Sets

In this part, we investigate which kind of usual sets encountered in abstract algebra are abstract cones such as the set of integers $\mathbb{Z}$, the set of odd numbers *O*, the set of even numbers *E*, the set of rational numbers *Q*, the set of irrational numbers Q^c, the set of non-negative real numbers $\mathbb{R}^+$, the set of real numbers $\mathbb{R}$, the set of complex numbers $\mathbb{C}$ and the set of $\mathbb{Z}_p$ where *p* is a prime number.

We discuss whether the following sets with usual addition form some algebraic structures or not.

Table (1)

	$\mathbb{Z}$	O	E	Q	Q^c	$\mathbb{R}^+$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Z}_p$
Semi-group	✓	×	✓	✓	×	✓	✓	✓	✓
Commutative Semi-group	✓	×	✓	✓	×	✓	✓	✓	✓
Monoid	✓	×	✓	✓	×	✓	✓	✓	✓
Commutative Monoid	✓	×	✓	✓	×	✓	✓	✓	✓
Abstract cone	✓	×	×	✓	×	✓	✓	✓	×

We discuss whether the following sets with usual multiplication form some algebraic structures or not.

Table (2)

	$\mathbb{Z}$	O	E	Q	Q^c	$\mathbb{R}^+$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Z}_p$
Semi-group	✓	✓	✓	✓	×	✓	✓	✓	✓
Commutative Semi-group	✓	✓	✓	✓	×	✓	✓	✓	✓
Monoid	✓	✓	×	✓	×	✓	✓	✓	✓
Commutative Monoid	✓	✓	×	✓	×	✓	✓	✓	✓
Abstract cone	✓	✓	×	✓	×	✓	✓	✓	×

Conclusion

Abstract cones are constructed on the abstract algebraic structure; need not be the convex as like the certain cones on the vector space.

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