Preservation properties of the moment generating function

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Abstract

The purpose of this paper is to study some preservation properties of the moment generating function, ordering of residual lives (mg-rl) under the reliability operations of mixture and convolution. Some examples of interest in reliability theory are also presented.

Keywords : Stochastic orders, Convolution, Mixture.

INTRODUCTION

Stochastic comparisons between probability distributions play a fundamental role in probability, statistics, and some related areas, such as reliability theory, survival analysis, economics, and actuarial science. One of these comparisons is the moment generating function order, that is recalled here.

Throughout the paper, the terms increasing and decreasing are used in the weak sense. All expectations and integrals are implicitly assumed to exist whenever they are written. Let X and Y be two non-negative random variables, representing equipment lives with distributions F_X and F_Y, and denote their survival functions by $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, respectively. Their moment generating functions are defined as, for all $s > 0$.

$$\psi_X(s) = \int_0^\infty e^{su} dF_X(u), \text{ and } \psi_Y(s) = \int_0^\infty e^{su} dF_Y(u).$$

Given two random variables X and Y, X is said to be smaller than Y in the moment generating function order (denoted by ($X \leq_{mg} Y$) if $E[e^{t_0 Y}]$ is finite for some $t_0 > 0$, and $\psi_X(s) \leq \psi_Y(s)$, for all $s > 0$.

Klar and Müller (2003) showed that if we denote

$$\psi_X^*(s) = \int_0^\infty e^{su} \bar{F}_X(u) du, \text{ and } \psi_Y^*(s) = \int_0^\infty e^{su} \bar{F}_Y(u) du,$$

then

$$X \leq_{mg} Y \Leftrightarrow \psi_X^*(s) \leq \psi_Y^*(s), \text{for all } s > 0.$$

Applications, properties and interpretations of the moment generating function order can be found in Kalar and Müller (2003), Zhang and Li (2003), Li (2004) and Ahmad and Kayid (2004).

Let $X_t = [X - t \mid X > t]$ and $Y_t = [Y - t \mid Y > t]$ denote the additional residual life at time t of the random lives X and Y; respectively for all $t \in (0, l_X) \cap (0, l_Y)$, where $l_X = sup\{t : F_X(t) < 1\}$ and $l_Y = sup\{t : F_Y(t) < 1\}$.

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A stronger comparison based on the moment generating functions of the distributions of *X* and *Y* has been recently defined in Wang and Ma (2009). They called it moment generating function of residual lives order, and its definition is the following.

A nonnegative random variable *X* is said to be smaller than *Y* in the moment generating of residual lives order (denoted by $(X \leq_{mg} Y)$ if

$$(X_t \leq_{mg} Y_t), \text{ for all } t \in (0, l_X) \cap (0, l_Y). \tag{1}$$

In the current investigation, we provide some preservation properties of the moment generating function of residual lives (mg-rl) order Section 1, contains definitions, notation and basic properties used through the paper. In section 2, we present some preservation results under the operations of convolution and mixtures. We also describe some sample examples of applications in recognizing situations where the random variables are comparable according to the moment generating function of residual life order.

PRELIMINARIES

Given two random variables *X* and *Y*, let us denote for all $s > 0$

$$\psi^*_{X_t}(s) = \frac{\int_t^\infty e^{su}\bar{F}_X(u)du}{e^{su}\bar{F}_X(t)}, \text{ and } \psi^*_{Y_t}(s) = \frac{\int_t^\infty e^{su}\bar{F}_Y(u)du}{e^{su}\bar{F}_Y(t)}.$$

Observe that, by the equation (1), it holds

$$X \leq_{mg-rl} Y \Leftrightarrow \psi^*_{X_t}(s) \leq \psi^*_{Y_t}(s), \text{ for all } t; s > 0.$$

Actually, an equivalent condition for mg-rl order is given in Wang and Ma (2009), and is as follows.

Proposition

Let *X* and *Y* two continuous non-negative random variables. Then

$$X \leq_{mg-rl} Y \Leftrightarrow \frac{\int_t^\infty e^{su}\bar{F}_X(u)du}{\int_t^\infty e^{su}\bar{F}_Y(u)du} \text{ is decreasing } t \in (0, l_X) \cap (0, l_Y), \text{ for all } s > 0.$$

We also recall the definitions of two notions that will be used in the next section.

Definition

A probability vector $\bar{\alpha} = (\alpha_1, ..., \alpha_n)$ with $\alpha_i > 0$ for all $i = 1, 2, ..., n$ is said to be smaller than the probability vector $\bar{\beta} = (\beta_1, ..., \beta_n)$ in the sense of discrete likelihood ratio order, denoted by $\bar{\alpha} \leq_{dlr} \bar{\beta}$,

$$\text{if } \frac{\beta_i}{\alpha_i} \leq \frac{\beta_j}{\alpha_j}, \text{ for all } 1 \leq i \leq j \leq n.$$

Definition

A function $g(x), -\infty < x < \infty$, is said to be a Polya function of order 2 (P F_2) if

(a) $g(x) \geq 0$ for $-\infty < x < \infty$, and

(b) $\begin{vmatrix} g(x_1 - y_1) & g(x_1 - y_2) \\ g(x_2 - y_1) & g(x_2 - y_2) \end{vmatrix} \geq 0$ for all $g(x), -\infty < x_1 < x_2 < \infty$ and $-\infty < y_1 < y_2 < \infty$, of

equivalently,

(b′) $\log[g(x)]$ is concave on $(-\infty, \infty)$.

The equivalence of (b) and (b′) is shown in Barlow and Proschan (1981).

PRESERVATION RESULTS

Preservation properties of an order under some reliability operations are of importance in reliability theory (Barlow and Proschan 1981). In this section, we give some preservation properties of the moment generating function ordering of residual lives (mg-rl). We begin by showing that the (mg-rl) order is preserved under convolutions, when appropriate assumptions are satisfied.

Theorem

Let X_1, X_2 and Y be three non-negative random variables, where Y is independent of both X_1 and X_2, and let Y have density g. If $X_1 \leq_{mg} X_2$ and g is log-concave then $X_1 + Y \leq_{mg} X_2 + Y$.

Proof: Because of Proposition 1.1, it is enough to show that for all $0 \leq t_1 \leq t_2$ and $x > 0$,

$$\frac{\int_0^\infty \int_{-\infty}^{t_1} e^{sx} P[X_1 \geq x + u] g(t_1 - u) du dx}{\int_0^\infty \int_{-\infty}^{t_1} e^{sx} P[X_2 \geq x + u] g(t_1 - u) du dx} \geq \frac{\int_0^\infty \int_{-\infty}^{t_2} e^{sx} P[X_1 \geq x + u] g(t_2 - u) du dx}{\int_0^\infty \int_{-\infty}^{t_2} e^{sx} P[X_2 \geq x + u] g(t_2 - u) du dx}$$

Since Y is non-negative then $g(t-u) = 0$ when $t < u$, hence the above inequality is equivalent to for all $0 \leq t_1 \leq t_2$, or equivalently

$$\begin{vmatrix} \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2} \geq (x+u) g(t_2 - u) du dx & \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1} \geq (x+u) g(t_2 - u) du dx \\ \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2} \geq (x+u) g(t_1 - u) du dx & \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1} \geq (x+u) g(t_1 - u) du dx \end{vmatrix} \geq 0 \qquad (2)$$

By the well-known basic composition formula (Karlin 1968,p. 17), the left side of (2) is equal to

$$\iint_{u_1 < u_2} \begin{vmatrix} g(t_2 - u_1) & g(t_2 - u_2) \\ g(t_1 - u_1) & g(t_1 - u_2) \end{vmatrix} \begin{vmatrix} \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2} \geq (x+u_1) du dx & \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1} \geq (x+u_1) du dx \\ \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2} \geq (x+u_2) du dx & \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1} \geq (x+u_2) du dx \end{vmatrix} du_1 du_2$$

The conclusion now follows if we note that the first determinant is non-positive since g is log-concave, and that the second determinant is non-positive since $X_1 \leq_{mg-rl} X_2$. This completes the proof.

Corollary

If $X_1 \le_{mg-rl} Y_1$ and $X_2 \le_{mg-rl} Y_2$ where X_1 is independent of X_2 and Y_1 is independent of Y_2, then the following statements hold:

(i) If X_1 and Y_2 have log-concave densities, then $X_1 + X_2 \le_{mg-rl} Y_1 + Y_2$.

(ii) If X_2 and Y_1 have log-concave densities, then $X_1 + X_2 \le_{mg-rl} Y_1 + Y_2$.

Proof: The following chain of inequalities, which establish (i), follows by Theorem 2.1:

$$X_1 + X_2 \le_{mg-rl} X_1 + Y_2 \le_{mg-rl} Y_1 + Y_2.$$

The proof of (ii) is similar.

Theorem

If $X_1, X_2, ...$ and $Y_1, Y_2, ...$ are sequences of independent random variables with $X_i \le_{mg-rl} Y_i$ and X_i, Y_i have log-concave densities for all i, then

$$\sum_{i=1}^{n} X_i \le_{mg-rl} \sum_{i=1}^{n} Y_i \quad (\text{n} = 1,2,...)$$

Proof: We shall prove the theorem by induction. Clearly, the result is true for $n = 1$.

Assume that the result is true for $n-1$, i.e.,

$$\sum_{i=1}^{n-1} X_i \le_{mg-rl} \sum_{i=1}^{n-1} Y_i \tag{3}$$

Note that each of the two sides of (3) has a log-concave density (see, e.g., Karlin 1968, p. 128). Appealing to (2.2) Corollary, the result follows.

Let $X_i, i = 1,...,n$ be a collection of independent random variables. Suppose that F_i and $\bar{F}_i$ are the distributions and survivals function of X_i, respectively. Let $\bar{\alpha} = (\alpha_1,...,\alpha_n)$ and $\bar{\beta} = (\beta_1,...,\beta_n)$ be two probability vectors. Let X and Y be two random variables having the respective survival functions $\bar{F}$ and $\bar{G}$ defined by

$$\bar{F}(x) = \sum_{i=1}^{n} \alpha_i \bar{F}_i(x) \text{ and } \bar{G}(x) = \sum_{i=1}^{n} \beta_i \bar{F}_i(x) \tag{4}$$

The following result gives conditions under which X and Y are comparable with respect to the (mg-rl) order. Actually, this is a closure under mixture property of the moment generating function of residual lives order.

Theorem

Let $X_1, ..., X_n$ be a collection of independent random variables with corresponding survival functions $\bar{F}_1, ..., \bar{F}_n$, such that $X_1 \le_{mg-rl} X_2 \le_{mg-rl} ... \le_{mg-rl} X_n$ and let $\bar{\alpha} = (\alpha_1,...,\alpha_n)$ and $\bar{\beta} = (\beta_1,...,\beta_n)$ such that $\bar{\alpha} \le_{dir} \bar{\beta}$. Let X and Y have survival functions $\bar{F}$ and $\bar{G}$ defined in (4). Then $X \le_{mg-rl} Y$.

Proof: Again because of Proposition (1.1), we need to establish that

$$\frac{\int_0^\infty e^{su}\sum_{i=1}^n \beta_i \bar{F}_i(x+u)du}{\int_0^\infty e^{su}\sum_{i=1}^n \alpha_i \bar{F}_i(x+u)du} \geq \frac{\int_0^\infty e^{su}\sum_{i=1}^n \beta_i \bar{F}_i(y+v)du}{\int_0^\infty e^{su}\sum_{i=1}^n \alpha_i \bar{F}_i(y+v)du}, \text{ for all } 0<y<x. \qquad (5)$$

Multiplying by the denominators and canceling out equal terms it can be shown that (5) is equivalent to

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ i\neq j}}^n \beta_i \alpha_j \int_0^\infty e^{su}\bar{F}_i(u+x)du \int_0^\infty e^{sv}\bar{F}_j(v+y)dv \leq \sum_{i=1}^n \sum_{\substack{j=1 \\ i\neq j}}^n \beta_i \alpha_j \int_0^\infty e^{su}\bar{F}_i(u+y)du \int_0^\infty e^{sv}\bar{F}_j(v+x)dv.$$

or

$$\sum_{\substack{i=1 \\ i<j}}^n \sum_{j=1}^n \left[\beta_i \alpha_j \int_0^\infty e^{su}\bar{F}_i(u+x)du \int_0^\infty e^{sv}\bar{F}_j \geq (v+y)dv + \beta_j \alpha_i \int_0^\infty e^{su}\bar{F}_j(u+x)du \int_0^\infty e^{sv}\bar{F}_i \geq (v+y)dv \right]$$

$$\leq \sum_{\substack{i=1 \\ i<j}}^n \sum_{j=1}^n \left[\beta_i \alpha_j \int_0^\infty e^{sv}\bar{F}_i(v+y)dv \int_0^\infty e^{su}\bar{F}_j(u+x)du + \beta_j \alpha_i \int_0^\infty e^{sv}\bar{F}_j(v+x)dv \int_0^\infty e^{su}\bar{F}_i(u+y)du \right]$$

Now, for each fixed pair (i, j) with $i<j$ we have

$$\left[\beta_i \alpha_j \int_0^\infty e^{sv}\bar{F}_i(v+y)dv \int_0^\infty e^{su}\bar{F}_j(u+x)du + \beta_j \alpha_i \int_0^\infty e^{sv}\bar{F}_j(v+y)dv \int_0^\infty e^{su}\bar{F}_i(u+y)du \right]$$

$$-\left[\beta_i \alpha_j \int_0^\infty e^{su}\bar{F}_i(u+x)du \int_0^\infty e^{sv}\bar{F}_j(v+y)dv + \beta_j \alpha_i \int_0^\infty e^{su}\bar{F}_j(u+x)du \int_0^\infty e^{sv}\bar{F}_i(v+y)dv \right]$$

$$=(\beta_i \alpha_j - \beta_j \alpha_i)\left[\int_0^\infty e^{sv}\bar{F}_i(v+y)dy \int_0^\infty e^{su}\bar{F}_j(u+x)dx - \int_0^\infty e^{su}\bar{F}_i(u+x)dx \int_0^\infty e^{sv}\bar{F}_j(v+y)dy \right].$$

which is non-negative because both terms are non-negative by assumption. This completes the proof.

To demonstrate the usefulness of the above results in recognizing (mg-rl)-ordered random variables, we consider the following examples.

Example

Let $X_{\bar{\lambda}}$ denote the convolution of n exponential distributions with parameters $\lambda_1,....,\lambda_n$ respectively. Assume without loss of generality that $\lambda_1 \leq .... \leq \lambda_n$. Since exponential densities are log-concave, Theorem 2.3 implies that $X_{\bar{\lambda}} \leq_{mg-rl} Y_{\bar{\mu}}$ whenever $\lambda_i \geq \mu_i$ for $i=1,...,n.$

Example

Let $X_i \sim Exp(\lambda_i), i=1,...,n$ be independent random variables. Let X and Y be $\bar{\alpha}$ and $\bar{\beta}$ mixtures of X_i's. An application of Theorem 2.4, immediately $X \leq_{mg-rl} Y$ for every two probability vector $\bar{\alpha}$ and $\bar{\beta}$ such that $\bar{\alpha} \leq_{dir} \bar{\beta}$.

Another application of Theorem 2.4 is contained in following example.

Example

Let $X_{\bar{\lambda}}$ and $X_{\bar{\mu}}$ be as given in Example 2.5. For $0 \le q \le p \le 1$ and $p+q=1,$ we have

$$pX_{\bar{\lambda}} + qX_{\bar{\mu}} \le_{mg-rl} qX_{\bar{\lambda}} + pX_{\bar{\mu}}$$

Acknoledgements

I would like to express my sincere gratitude to Dr. Tin Htway, Rector of Hinthada University, and Dr. Theingi Shwe, Pro-Rector of Hinthada University, for their kind permission and encouragement to carry out this research. I am grateful to Dr. Kyaw Zaw Oo (Head of the Department of Mathematics, Hinthada University) for his critical reading of the manuscript and valuable advices.

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