A Way to Solve System of Linear Equations by Means of Norm of Linear Mapping

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Abstract

In this paper, we study a way to solve system of linear equations by means of norm of contionuous linear mapping. Firstly, some properties of continuous linear mapping includes that every continuous linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ can be represented as a finite matrix of real numbers and the inverse operator method is described. Then two ways of computing norm of a square matrix is examined. Finally, some pratical problems are solved by this inverse operator method by using norm of continuous linear operator. Before computing pratical problems like $BX = C, B$ is transformed into $I + A,$ where $\|A\| < 1.$ Then the solution is $(I + A)^{-1}C$.

Keywords: norm of contionuous linear mapping, L- norm and M- norm of matrix, inverse operator method.

Introduction

There are several ways to solve system of linear equations such as Gauss Elimination method, Gauss Seidel method and Gauss Jordan method as well as Cramer's Rule. But, there are many rare methods to solve these equations by means of norms of continuous linear mapping.

The research paper is organized as three main parts. The first one concerns with properties of continuous linear mapping. The second part involves L- norm and M- norm of matrix and the inverse operator method using norm of continuous linear mapping. Finally the third part regards practical computing of system of linear equations by means of inverse operator method.

Properties of Continuous Linear Mapping

In this part, properties of continuous linear mapping is described by some definitions and theorems.

Definition.[3] Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be defined by $Tx = Ax$, where A is an $m \times n$ matrix of real numbers. Then A is a continuous linear mapping.

Definition.[3] Let X and Y be real normed linear spaces. Then we define

$L(X,Y) = \{T \mid T \text{ is a map} : X \to Y \text{ be linear and continuous}\}$.

Theorem.[3] Suppose $T:\mathbb{R} \to \mathbb{R}$ is a linear mapping if and only if there exists $a \in \mathbb{R}$ such that $T(x) = a\,x$ for all $x \in \mathbb{R}$.

Proof: Let $\{x\}$ be a basis of $\mathbb{R}$.

Then for $y \in \mathbb{R}$, there exists $\lambda \in \mathbb{R}$ such that $y = \lambda\, x.$

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So $T(y) = T(\lambda x) = \lambda T(x)$.

Since $T(x) \in \mathbb{R}$, *there exists* $\mu \in \mathbb{R}$ *such that* $T(x) = \mu x$.

So, $T(y) = \lambda(\mu x)$

$= \mu(\lambda x)$

$= \mu y.$

Choose $a = \mu$. Then $T(y) = a\, y.$

Then the linear mapping T can be defined $T(y) = a\, y.$

Suppose $T(y) = a\, y$ for all $y \in \mathbb{R}$.

Then we will prove that T is linear.

For $\alpha,\ \beta \in \mathbb{R},\ \ x,\ y \in \mathbb{R},\ T(\alpha x + \beta y) = a(\alpha x + \beta y)$

$= \alpha\, a\, x + \beta\, a\, y$

$= \alpha\, T(x) + \beta\, T(y).$

So, T is linear.

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $Tx = Ax$, where $x \in \mathbb{R}^n$ and A is an $m \times n$ matrix of real numbers. Then T is linear if and only if there is a $m \times n$ matrix $A = (a_{ij})$ and $Tx = Ax$.

Proof: See [3].

Norms of a Matrix and Inverse Operator Method

In this part, we state L-norm, M-norm of a matrix and some theorems for inverse operator method are studied.

Definition.[4] Let $A = (a_{ij})$ where $a_{ij} \in \mathbb{R}$, $1 \le i \le m,\ 1 \le j \le n$. Then we define

$\|A\|_L = \max \sum_i |a_{ij}|$ **(L-norm)** and

$\|A\|_M = \max \sum_j |a_{ij}|$ **(M-norm)** .

For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\|A\|_L = \max\{1+2, 3+4\} = \max\{3,7\} = 7,$

$\|A\|_M = \max\{1+3,\ 2+4\} = \max\{4,6\} = 6.$

Theorem. Let X be a real Banach space and the continuous and linear mapping A from X into X *with* $\|A\| < 1.$

Then $(I + A)^{-1} = I - A + A^2 - A^3 + A^4 + \cdots (-1)^n A^n + \cdots = \sum_{n=0}^{\infty} (-1)^n A^n$ denoting $A^0 = I$

Proof: This theorem can be proved in two ways. See [1] or [2].

Theorem. If V is a real Banach Algebra with unity e and $x \in V,\ \|x\| < 1,$ then

$$(e+x)^{-1} = e - x + x^2 - x^3 + \cdots (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n,\ x^0 = e.$$

We use the fact that $L(X,\ X)$ is a Banach Algebra with identity I as unity, under the operation of composition of functions.

Proof: See [1].

Theorem (Inverse Operator Method). Let $(I+A)X = B$ where A is a $n \times n$ matrix, X is a $n \times 1$ matrix, B is a $n \times 1$ matrix and I is an identity $n \times n$ matrix.

If $\|A\| < 1$, then $(I+A)^{-1} = I - A + A^2 - A^3 + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n A^n, I = A^0.$$

Hence X can be evaluated by

$$X = (I+A)^{-1}\, B = (I - A + A^2 - A^3 + \cdots)\, B\,.$$

Proof: See [4].

Practical Computing by Inverse Operator Method

In this part, some practical problems are solved by inverse operator method using norms of a matrix.

Example. We will solve the following linear system by $(I+A)^{-1}$ method,

$$x_1 + \frac{1}{5}x_2 + \frac{1}{3}x_3 = 0,$$

$$\frac{1}{4}x_1 + x_2 + \frac{1}{5}x_3 = 1,$$

$$\frac{1}{4}x_1 + \frac{1}{5}x_2 + x_3 = 0 \text{ in } M_{3\times3}(\mathbb{R}).$$

We may write it in the form $(I+A)X = C,$ So we have

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & 0 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where $A=\begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & 0 \end{pmatrix}$ and $\|A\| = \max\left\{\frac{1}{5}+\frac{1}{3},\ \frac{1}{4}+\frac{1}{5},\ \frac{1}{4}+\frac{1}{5}\right\} = \frac{8}{15} < 1.$

Then $X = (I+A)^{-1}C$

$$= C - AC + A^2C - A^3C + \cdots + (-1)^n A^n C + \cdots$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} + \begin{pmatrix} \frac{1}{15} \\ \frac{9}{100} \\ \frac{1}{20} \end{pmatrix} - \begin{pmatrix} \frac{13}{375} \\ \frac{2}{75} \\ \frac{13}{375} \end{pmatrix} + \begin{pmatrix} \frac{19}{1125} \\ \frac{117}{7500} \\ \frac{23}{1825} \end{pmatrix} - \cdots .$$

So, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -0.15111 \\ 1.0789333 \\ -0.170666 \end{pmatrix}$.

Now, the approximate solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -0.15111 \\ 1.0789333 \\ -0.170666 \end{pmatrix}$.

Regarding the use $(I+A)^{-1}$ Method to solve equations in $M_{n\times n}(\mathbb{R})$, the condition $\|A\|<1$ may seem to be too restrictive to be useful in practice. However, at least for some matrices A in $M_{n\times n}(\mathbb{R})$, we can find $k \in \mathbb{R}$ and $B \in M_{n\times n}(\mathbb{R})$ such that $A = k(I+B)$ and $\|B\|<1$.

For instance, let us consider the following system

$$125x_1 + 18x_2 + x_3 = 12,$$

$$13x_1 - 87x_2 + 62x_3 = 175,$$

$$28x_1 + 14x_2 + 67x_3 = 97.$$

Multiplying the second equation by (-1) and dividing all equations by 100, we get

$$1.25x_1 + 0.18x_2 + 0.01x_3 = 0.12,$$

$$-0.13x_1 + 0.87x_2 - 0.62x_3 = -1.75,$$

$$0.28x_1 + 0.14x_2 + 0.67x_3 = 0.95.$$

$$\text{Then} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0.25 & 0.15 & 0.01 \\ -0.13 & -0.13 & -0.62 \\ 0.28 & 0.14 & -0.33 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.12 \\ -1.75 \\ 0.95 \end{pmatrix},$$

$$\textit{and} \text{ so } (I+B)X = C\,.$$

Here, $\|B\| = \max\{0.66, 0.45, 0.96\} = 0.96 < 1.$

Hence, $(I+B)^{-1}$ exists.

In general, given an equation $Tu = f$ where $T \in L(X,X)$, it will be worth of investigating under what conditions there exists $A \in L(X,X)$ and $k \in \mathbb{R}$ such that $T = k(I+A)$, $\|A\| < 1$.

Example. In an Electrical power distribution department, an engineer has a problem to distribute electricity among villages. For some villages, he wants to distribute 1.5 M-Watts to each village and for the remaining villages he wants to distribute 0.25 M-Watts to each villages. Then the total amount of electricity is 16 M-Watts. If he wants to distribute 0.25 M-Watts instead of 1.5 M-Watts for each village and he also distribute 1.5 M-Watts instead of 0.25 M-Watts, then the total amount of electricity will be 6 M-Watts. How will he distribute the electricity among villages and how many villages must be distributed?

We can solve this problem as follows:

$$1.5\, x_1 + 0.25\, x_2 = 16$$
$$0.25\, x_1 + 1.5\, x_2 = 6.$$

We may write it in the form in $M_{2\times 2}(\mathbb{R})$.

$$\begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 16 \\ 6 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 16 \\ 6 \end{pmatrix}$$

$(I+A)\,X = C$, where $A = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix}$ *and*

$\|A\| = \max\{0.75,\ 0.75\} = 0.75 < 1.$

So, $(I+A)^{-1}$ method can be used in solving the problem.

Hence X can be evaluated by $X = (I + A^{-1})C = (I - A + A^2 - A^3 + \cdots)C$, we have the solution.

Therefore x_1 = 10.2857 and x_2 = 2.285.

So, the engineer will distribute the electricity among 12 villages.

This method can be used in solving the functional equation $Tu = f$ provided there exists a real Banach space X and continuous linear operator A from X into X such that $\|A\| < 1$ and $T = I + A$, where I *is* the identity mapping.

Conclusion

A way to solve system of linear equations by means of inverse operator method using norm of a continuous linear mapping has been presented.

This way is available to solve linear ordinary differential equations and linear partial differential equations as well as Fredholun-integral equations on a special Sobolev space $H^m(\Omega)$ where Ω is an open set in the Euclidean space $\mathbb{R}^n$.

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