Flow Past an Ellipse

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Abstract

Flows past an ellipse are sketched by using MATLAB software to visualize the flow patterns. It is also studied by using the conformal transformation for various angles of inclination with the x–axis.

Keywords: analytic functions, streamlines, transformation, visualization

Some Preliminary Concepts

Irrotational flow pattern around body of flap plate shape is the subject of this paper. It will be assumed that the fluid is inviscid and incompressible and the motion 2–dimensional. The 2–dimensional incompressible continuity equation guarantees the existence of a stream function ψ, from which the velocity components can be derived as

$$u = -\frac{\partial\psi}{\partial y}, \; v = \frac{\partial\psi}{\partial x}. \qquad \text{———(a)}$$

For irrotational motion the stream function satisfies Laplace's equation: $\underline{\nabla}^2\psi = 0$ (O'Neill, M.E & Chorlton, F.,1986). Likewise, the condition of irrotationality guarantees the existence of another scalar function ϕ,called the velocity potential, which is related the velocity components by

$$u = -\frac{\partial\phi}{\partial x}, \; v = -\frac{\partial\phi}{\partial y}. \qquad \text{———(b)}$$

And potential function satisfies Laplace's equation:$\underline{\nabla}^2\phi=0$.

Comparing (a) and (b) we obtain

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \; -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}.$$

These are Cauchy–Riemann condition for $\phi(x,y)+ i\psi(x,y)$ to be analytic function of the complex variable x+iy=z and so we can put

$$\phi(x,y)+ i\psi(x,y) = f(x+iy)=f(z),$$

where f is analytic function of z (O'Neill, M.E & Chorlton, F.,1986).

These facts can be useful for analyzing 2–dimensional potential flow for certain kinds of the boundary conditions. If we can find a solution of Laplace's equation for a simple boundary in z–plane, we can apply any analytic mapping we choose, and map the boundary and streamlines to another complex plane. Since the streamlines conformed to the boundary in the original plane, they automatically conform to the transformed boundary in the transformed plane, and since the mapping is analytic, the transformed streamlines are solution of Laplace's equation, just as were the streamlines in the original plane. Likewise the velocity potential maps from the original to the transformed plane(Ba Kyi,1976).

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In this paper we used conformal transformations. Conformal transform is a useful tool in the study of potential flow. The flow pattern (and hence velocity and pressure distribution) can be determined around or over a simple body; a conformal transformation can then change the shape of the body into a more complicated geometry, and the transformation also determines the flow pattern over the new body (Milne-Thomson L.M., 1968).

Consider the mapping of a region to z–plane to a region to Z–plane by the transform z = f(Z). If f(Z) is an analytic function of Z, then the mapping z = f(Z) is said to be conformal at all points where $f'(Z) \neq 0$. An interesting property of the conformal mapping is that f conserves the angle of intersection of two curves intersecting at $z = z_0$, $f'(z_0) \neq 0$.

Application of conformal mapping to simple fluid flows

Let z represent the physical plane in which the object shape is complicated and Z the transformed plane in which the object shape is simple and for which the complex potential w(Z) is known.

Consider the analytic mapping function Z=f(z) with $\nabla^2 X = \nabla^2 Y = 0$, where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and the Cauchy Riemann conditions, $X_x = X_y$ and $X_y = -Y_x$.

$$\phi_x = \phi_X X_x + \phi_Y Y_x$$

$$\phi_{xx} = \phi_{XX} X^2_x + \phi_X X_{xx} + 2\phi_{xy} X_x Y_x + \phi_Y Y_{xx} + \phi_{YY} Y^2_x$$

$$\nabla\phi = \phi_{xx} + \phi_{yy} (=0) = (Y^2_x + Y^2_y)(\phi_{YY} + \phi_{XX})$$

Hence $\phi_{XX} + \phi_{YY} = 0$. Similarly $\psi_{XX} + \psi_{YY} = 0$.

It follows that w is complex potential in both z and Z planes provided Z=f(z) (equivalently, z=g(Z)) is an analytic transformation (Churchill, Ruel V., Brown, James W., Verhery, Rager F., 1974).

Joukowski Transformation

One of the most important conformal transformations is that known as Joukowski transformation (Wilson,D.H, 1964). The Joukowski transformation is defined to be

$$z = Z + \frac{a^2}{Z}, \; a > 0$$

where $z = x+iy$ or $re^{i\theta}$ and $Z = X + iY$ or $Re^{i\gamma}$, x ,y, r, θ, X, Y, R, γ are all real.

Application of Joukowski Transformation to Aerofoil

Consider Joukowski transformation applied to a circle of radius c in the Z–plane at the point $Z = Z_0$. Then the circle $X^2 + Y^2 = c^2$ in the Z–plane transformed into the ellipse $\frac{x^2}{1+\frac{a^2}{c^2}} + \frac{y^2}{1-\frac{a^2}{c^2}} = c^2$ in z–plane. The ellipse in the physical plane has major axis, $\ell = c + \frac{a^2}{c}$ and a minor axis, $m = c - \frac{a^2}{c}$. When $\frac{m}{\ell}$ is small the ellipse is thin. At the limit $m \to 0$, it can be seen $a \to c$ and $\ell \to 2c$. Hence we can map a flat plate of length 4c to a circle of radius c.

Hence the concept of transformation has been developing geometrics in the transformed plane more complicated than in physical plane (Milne-Thomson L.M., 1968).

Flow around an ellipse

The Joukowski transformation $z = Z + \frac{a^2}{Z}$, $a > 0$ transforms the circle $|Z| = a>c$ in the Z-plane into the ellipse $x = (c+\frac{a^2}{c})\cos\theta$, $y = (c - \frac{a^2}{c})\sin\theta$ in the z-plane. For an ellipse with major and minor semi axis A, B respectively, the constants a, c are $c = ½(A +B)$, $a^2 = ¼(A^2 – B^2)$. Conversely, the transformation $Z = ½(z+\sqrt{z^2 - 4a^2})$ maps the region consisting of the exterior and periphery of the ellipse $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ in z-plane onto the region comprising of the exterior and periphery of the circle $x^2+y^2 = a^2$ in Z-plane.

It therefore follows that the complex potential $w(Z)=U(Ze^{-i\alpha} + \frac{a^2}{Z}e^{i\alpha})$ together with the transformation with the transformation $Z = (½(z+\sqrt{z^2 - 4a^2})$ will give the complex potential $w(z) = ½U(z+\sqrt{z^2 - 4a^2})\, e^{-i\alpha} + 2Uc^2e^{i\alpha}\, ½(z+\sqrt{z^2 - 4a^2})^{-1}$ which describes the flow of a uniform stream of speed U past an ellipse when the direction of the stream make an angle α with major axis of the ellipse.

Consider the complex potential in Z–plane for flow past a cylinder at an angle α to X–axis.

$$w(Z) = U(Ze^{-i\alpha} + \frac{a^2}{Ze^{-i\alpha}}) .$$

As $Z\to \infty$, $w(Z) \to UZe^{-i\alpha}$ which is a uniform flow.

The stagnation points of this flow are obtained by putting $\frac{dw}{dZ}=0$.

$$\frac{dw}{dZ} = U(e^{-i\alpha} + \frac{a^2}{Z^2e^{-i\alpha}}) = 0$$

$$Z^2 = a^2(e^{2i\alpha})^2$$

$$Z = ae^{i\alpha} ,\ ae^{i(\alpha+\pi)}$$

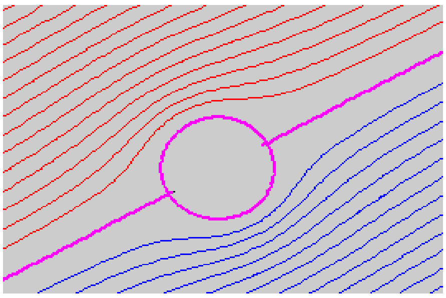

Figure (1). Flow around an incident circle

By the transformation the circle $Z=ce^{i\theta}$ in Z–plane is mapped into the ellipse in z–plane. The potential for large z is mapped according to Z=z and then $w(Z)=w(z)=Uze^{-i\alpha}$ which corresponds to the uniform flow at angle α in the z–plane. A sketch of the mapped flow is shown in the z–plane. The complex potential for the flow past an ellipse at an angle α to the x–axis is

$$w(z) = U(\tfrac{1}{2}(z+\sqrt{z^2-4c^2})e^{-i\alpha} + \frac{2a^2 e^{i\alpha}}{z+\sqrt{z^2-4c^2}})$$

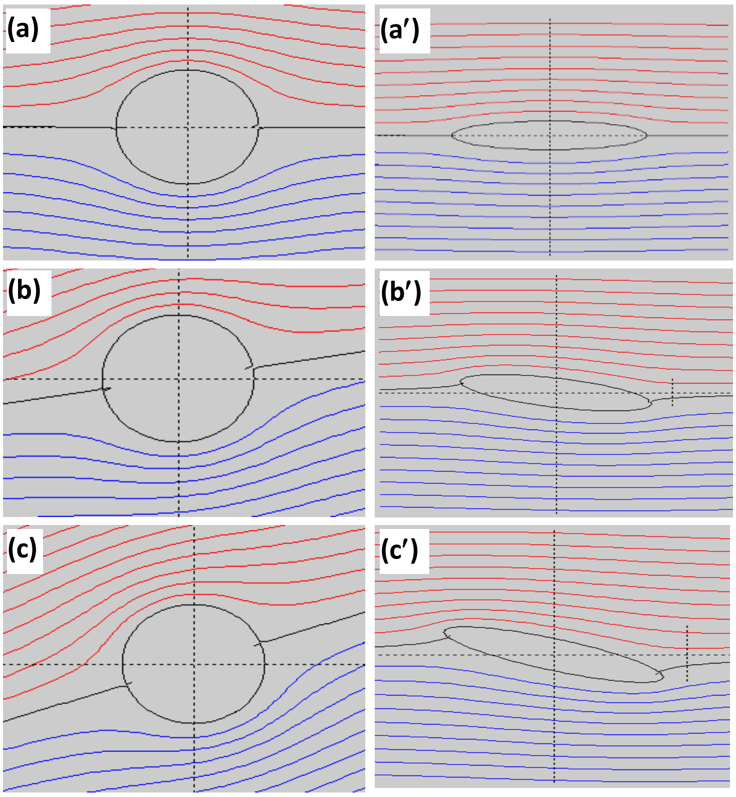

Figure (2). Mapped flow past the circles in Z-plane (Figures a, b, and c) to the flow past an ellipse the z-plane (Figures a′, b′, and c′).

The flow pattern with various angle of attack are symmetry. From figure (a′), stagnation points coincide with leading edge and trailing edge. From figure (b′) and (c′), stagnation points are located on the elliptical body near the edges on the major-semi axis of the body. At the stagnation points the velocity vanishes. Consequently, high pressure occurs at front side and rear of the ellipse, since streamlines are loose around stagnation points.

We can conclude, although flow past an ellipse in z-plane is mapped flow from flow past a circle, streamlines are parallel to the line of symmetric body. Hence there is no lift.

Flow with circulation past an ellipse

Consider we add a circulation to the complex potential in Z–plane for flow past a cylinder at an angle α to X–axis.

$$w(Z) = U(Ze^{-i\alpha} + \frac{a^2}{Ze^{-i\alpha}} + \frac{ik}{2\pi}\ln Z)\,.$$

A sketch of the mapped flow is shown in the z–plane. The complex potential for the flow with circulation past an ellipse at an angle α to the x–axis in z–plane is

$$w(z) = U(\tfrac{1}{2}(z+\sqrt{z^2-4c^2})e^{-i\alpha} + \frac{2a^2 e^{i\alpha}}{z+\sqrt{z^2-4c^2}} + \frac{ik}{2\pi}\ln(z+\sqrt{z^2-4c^2}))$$

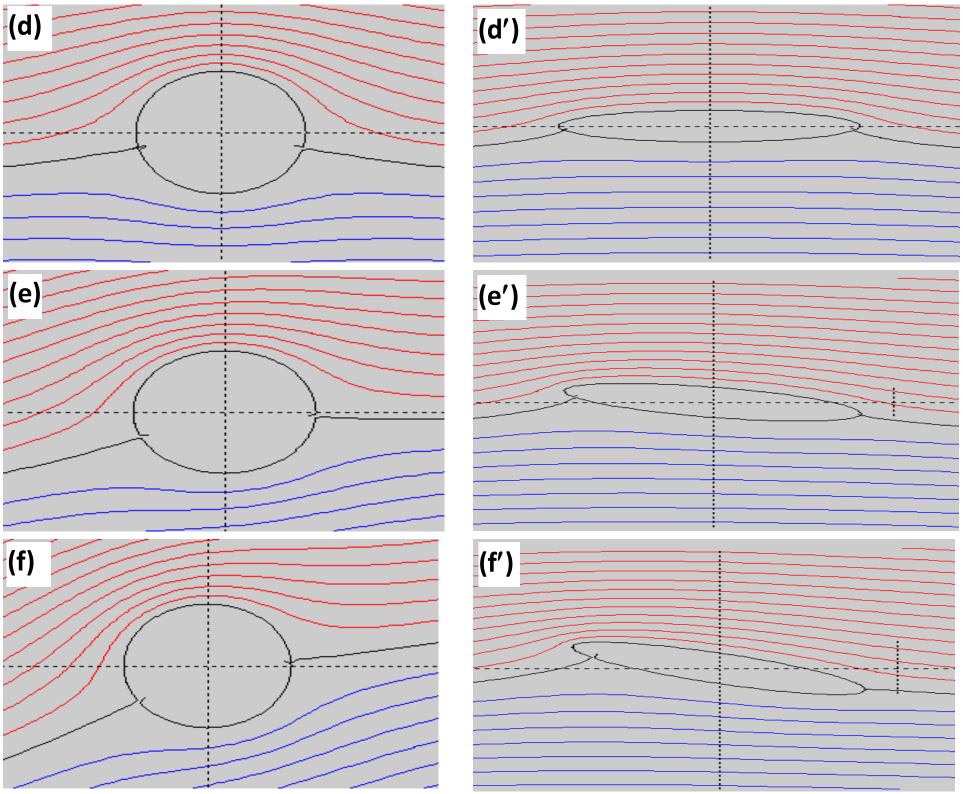

Figure (3). Mapped circulation flow past the circles in Z-plane (Figures d, e, and f) to the flow past an ellipse the z-plane (Figures d′, e′, and f′).

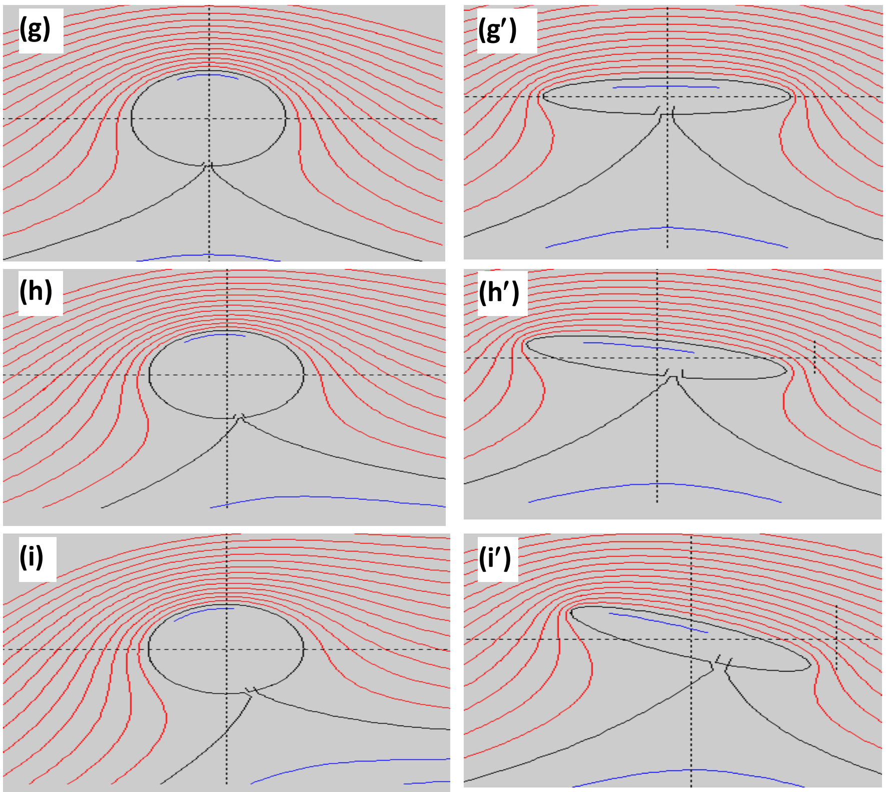

Figure (4). Mapped circulation flow past the circles in Z-plane (Figure g, h, and i) to the flow past an ellipse the z-plane (Figures g′, h′, and i′).

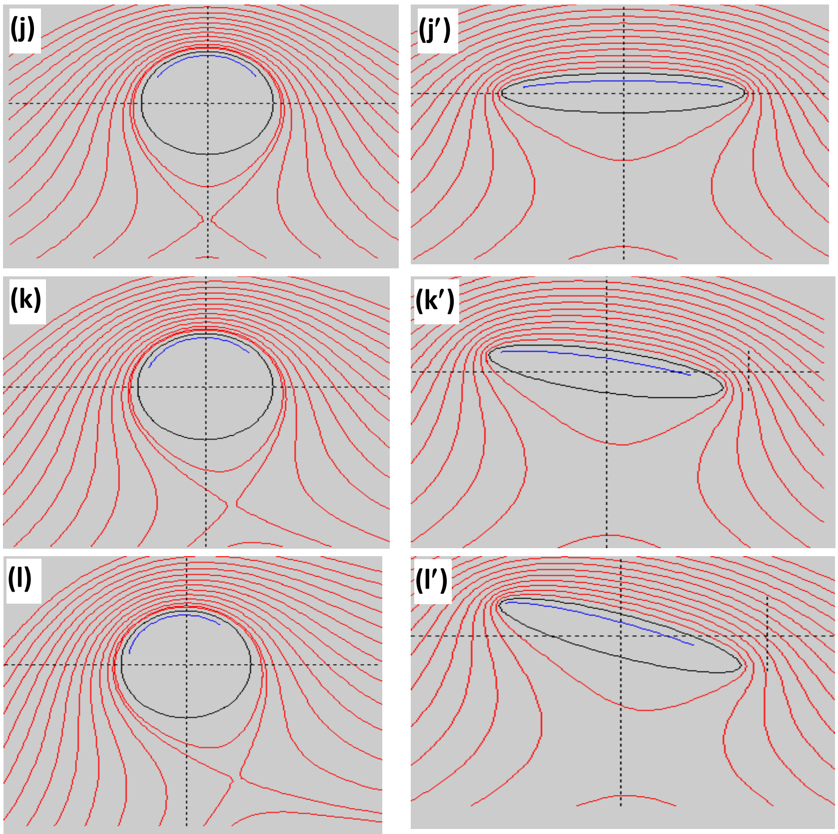

Figure (5). Mapped circulation flow past the circles in Z-plane (Figures j, k, and l) to the flow past an ellipse the z-plane (Figures j′, k′, and l′).

If we add the circulation to the above flow, font stagnation point moves a little form leading edge and rear stagnation point coincides with trailing edge. Then the leading pressure is low and the trailing pressure is high. At that time the body get lift. If incident angle with the flow, the body gets larger more lift. If we change the strength of circulation, the body also get lift. Here for convenience, incident angle has been taken between 0 and 22 degree. We take strength of circulation between -2 and 2.

Conclusion

Our visualization are troubles. One of the troubles with conformal mapping methods is that the above equation relate the velocity on the airfoil to the velocity on the circle, but the relation is not possible when $Z= \ell$. However, we know that the velocity on the airfoil does not go to infinity anywhere. Today with the advent of computers, that problem can be so easy using visualization method. There is an example of one result that was get with the utilization of the transformation. The experiment is designed to introduce to some of the techniques and approaches used in geometric circle. It also gains experience in observing flows and drawing conclusions about them from the observed behavior.

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