Direct Products and Direct Sums of Modules

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Abstract

The objects of study in this paper are modules over arbitrary rings, and they can be thought of as generalizations of vector spaces and abelian groups. Firstly, some basic definitions, including those of a module and a module homomorphism were introduced. Finally, the theorems and examples of direct product and direct sum on module homomorphism were proved.

Keywords: vector space, abelian group, additive subgroup, ring theorem

INTRODUCTION

Modules are central to the study of commutative algebra and homological algebra. Moreover, they are used widely in algebraic geometry and algebra. In a vector space, the scalars taken from a field act on the vectors by scalar multiplication, subject to certain rules. In a module, the scalars only need belonging to a ring, so the concept of a module is a significant generalization. Much of the theory of modules is concerned with extending the properties of vector spaces to modules. However, module theory can be much more-complicated than that of vector spaces.

Basic Concepts of a Module

Definition.[1] Let R be a ring, and let M be an abelian group. Then M is called a **left R-module** if there exists a scalar multiplication $\mu : R \times M \to M$ denoted by $\mu(r, m) = rm$, for all $r \in R$ and $m \in M$, such that

(i) $r(m_1 + m_2) = rm_1 + rm_2$ Distributivity

(ii) $(r_1 + r_2)m = r_1m + r_2m$ Distributivity

(iii) $r_1(r_2m) = (r_1r_2)m$ Associativity

(iv) $1.m = m$ Identity.

The fact that the abelian group M is a left R-module will be denoted by ${}_RM$.

Let A be an R-module and B a nonempty subset of A. Then B is called a **submodule** of A if (i) B is an additive subgroup of A (ii) $b \in B,\ r \in R \Rightarrow rb \in B$.

If B is a submodule of an R-module A, then A/B together with the operations $(x+B)+(y+B)=(x+y)+B$ and $(x+B)r = xr + B$ for $x+B,\ y+B \in A/B$ and $r \in R$ is called the **factor module** or the **quotient module** of A.

Let A and B be R-modules. A mapping $f : A \to B$ is called a **module homomorphism**, if for all $a, b \in A$, $r \in R$, for $f(a+b) = f(a) + f(b)$ and $f(ra) = rf(a)$.

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Remarks

(i) If R is a division ring, then f is called a linear transformation.

(ii) If f is injective, then f is called a monomorphism.
(iii) If f is surjective, then f is called an epimorphism.
(iv) If f is bijective, then f is called an isomorphism.
(v) A homomorphism from a group G to itself is called an endomorphism of G.
(vi) An isomorphism from a group G to itself is called an automorphism of G.

Theorem.[2] (Fundamental homomorphism theorem)

Let M and N be left R-modules. If $f : M \to N$ is any R-homomorphism, then $f(M) \cong M/\ker(f)$.

Proof: Let R be a ring, M and N be left R-module, let $f \in \mathrm{Hom}_R(M, N)$.

Thus $f : M \to N$ is an R-homomorphism.

Moreover, ker (f) $= \{x \in M / f(x) = 0\}$.

Define a mapping $\varphi : M/\ker(f) \to f(M)$ by $\varphi(m + \ker(f)) = f(m)$ for all $m \in M$.

Take any $m_1, m_2 \in M$, $m_1 + \ker(f) \in M/\ker(f)$ and $m_2 + \ker(f) \in M/\ker(f)$

$$m_1 + \ker(f) = m_2 + \ker(f)$$

$$m_1 - m_2 \in \ker(f)$$

$$f(m_1 - m_2) = 0$$

$$f(m_1) = f(m_2)$$

So φ is well-defined.

Consider $\varphi((m_1 + \ker(f)) + (m_2 + \ker(f))) = \varphi((m_1 + m_2) + \ker(f))$

$$= f(m_1 + m_2) = f(m_1) + f(m_2)$$

$$= \varphi(m_1 + \ker(f)) + \varphi(m_2 + \ker(f))$$ Again,

$\varphi(r(m_1 + \ker(f))) = \varphi(r m_1 + \ker(f)) = f(rm_1) = r\varphi(m_1 + \ker(f))$, $r \in R$.

Therefore φ is R-homomorphism.

Suppose $\varphi(m_1 + \ker(f)) = \varphi(m_2 + \ker(f))$

$$f(m_1) = f(m_2)$$

$$f(m_1 - m_2) = 0$$

Thus $m_1 - m_2 \in M$ and $m_1 - m_2 \in \ker(f)$, $m_1 + \ker(f) = m_2 + \ker(f)$. So φ is injective.
Take any $x \in f(M)$, $m^* \in M$ such that $f(m^*) = x$.

$m^* + \ker(f) \in M/\ker(f)$ and $\varphi(m^* + \ker(f)) = f(m^*) = x$.

Thus φ is surjective. Hence φ is isomorphism from f(M) onto $M/\ker(f)$.

That is $f(M) \cong M/\ker(f)$. ■

Theorem. [3] Let A be an R-module, I be a submodule of ker (f), $f : A \to \mathcal{S}$ be an R-module homomorphism and let $\overline{f} : A/I \to \mathcal{S}$ be defined by $\overline{f}(a+I) = f(a)$ for every $a \in A$. Then

(i) $\overline{f}$ is a unique homomorphism, Im $\overline{f}$ = Im f and $\ker(\overline{f}) = \ker(f)/I$.

(ii) $\overline{f}$ is an isomorphism if and only if $I = \ker(f)$ and f is an epimorphism.

(iii) $A/\ker(f) \cong \operatorname{Im} f$.

Proof: See [3]. ■

Finitely Generated Submodules

Definitions.[3] If X is a subset of a module A over a ring R, then the intersection of all submodules of A containing X is called **the submodule generated by X** (or **spanned by X**).
(i) If X is finite, and X generates the module A, A is said to be **finitely generated.**
(ii) If $X = \phi$, then X clearly generates the zero module.
(iii) If X consists of a single element, $X = \{a\}$, then the submodule generated by X is called **the cyclic (sub) module** generated by a.

If A is an R-module and $\{B_i \mid i \in I\}$ is a family of submodules of A, then the submodule generated by $X = \bigcup_{i \in I} B_i$ is called the **sum of the modules** B_i .If the index set I is finite, the sum of $B_1, B_2, ..., B_n$ is denoted by $B_1 + B_2 + ... + B_n$.

Theorem.[3] Let R be a ring, A an R-module, X a subset of A and $\{B_i \mid i \in I\}$ be a family of submodule of A. Then the sum of $\{B_i \mid i \in I\}$ consists of all finite sums
$b_{i_1} + b_{i_2} + ... + b_{i_n}, b_{i_k} \in B_{i_k}$, i.e, sum of

$$\{B_i \mid i \in I\} = \left\{ \sum_{k=1}^{n} b_{i_k} \mid n \in N^*, i_k \in I, b_{i_k} \in B_{i_k} \right\}.$$

Proof: Sum of $\{B_i \mid i \in I\}$ = the submodule of A generated by

$$X = \cup B_i = \left\{ \sum_{i=1}^{s} r_i a_i + \sum_{j=1}^{t} n_j b_j \mid s, t \in N^*, a_i, b_j \in X, r_i \in R, n_j \in Z \right\}.$$

Therefore each $a_i \in B_i$ for some $i \in I$ and since B_i is a submodule of A, we have $r_i a_i \in B_i$.
Similarly each $b_j \in B_j$ for some $j \in I$ and since B_j is a submodule of A, we have $n_j b_j \in B_j$.
Therefore the sum of $\{B_i \mid i \in I\}$ consists of all finite sums $b_{i_1} + b_{i_2} + ... + b_{i_n}$ with $b_{i_k} \in B_{i_k}$. ■

Definitions.[3] Let $\{A_i \mid i \in I\}$ be a family of nonempty sets. The (external) **direct product** $\prod_{i \in I} A_i$ **of that family is defined as** $\prod_{i \in I} A_i = \{\{a_i \mid i \in I\} \mid a_i \in A_i \text{ for every } i \in I\}$**.** We can write $\{a_i \mid i \in I\}$ by $\{a_i\}$ and define addition on $\prod_{i \in I} A_i$ by

$$\{a_i\} + \{b_i\} = \{a_i + b_i\} \text{ for } \{a_i\}, \{b_i\} \in \prod_{i \in I} A_i .$$

The (external) **direct sum** $\sum_{i\in I} A_i$ of the family $\{A_i\}_{i\in I}$ is defined as

$$\sum_{i\in I} A_i = \left\{\{a_i \mid i\in I\} \mid \{a_i\}\in \prod_{i\in I} A_i, a_i \neq 0_{A_i} \text{ for finitely many } i\right\}.$$

Thus $\sum_{i\in I} A_i \subset \prod_{i\in I} A_i$.

Theorem.[3] Let R be a ring R and $\{A_i \mid i\in I\}$ be a nonempty family of R-modules. $\prod_{i\in I} A_i$ is the direct product of the abelian group A_i and $\sum_{i\in I} A_i$ the direct sum of the abelian groups A_i.

Then

(i) $\prod_{i\in I} A_i$ is an R-module with the action of R given by $r\{a_i\} = \{ra_i\}$.

(ii) $\sum_{i\in I} A_i$ is a submodule of $\prod_{i\in I} A_i$.

(iii) For each $k\in I$, the canonical projection $\pi_k : \prod_{i\in I} A_i \to A_k$ is an R-module epimorphism.

(iv) For each $k\in I$, the canonical injection $I_k : A_k \to \sum_{i\in I} A_i$ is an R-module monomorphism.

Proof: (i) Take any $\{a_i\},\{b_i\}\in \prod_{i\in I} A_i$ and $r,s\in R$.

Then
$$\begin{aligned}
r(\{a_i\}+\{b_i\}) &= r\{a_i+b_i\} \\
&= \{r(a_i+b_i)\} \\
&= \{ra_i + rb_i\} \\
&= \{ra_i\}+\{rb_i\} \\
(r+s)\{a_i\} &= \{(r+s)a_i\} \\
&= \{ra_i + sa_i\} \\
&= \{ra_i\}+\{sa_i\} \\
&= r\{a_i\}+s\{a_i\} \\
(rs)\{a_i\} &= \{(rs)a_i\} \\
&= \{r(sa_i)\} \\
&= r\{sa_i\} \\
&= r(s\{a_i\}) .
\end{aligned}$$

Therefore $\prod_{i\in I} A_i$ is an R-module under the above operations.

(ii) Since $\{0_{A_i}\}\in \sum_{i\in I} A_i$, we have $\sum_{i\in I} A_i \neq \phi$.

Let $\{a_i\},\{b_i\} \in \sum_{i\in I} A_i$ and $r \in R$.

If $I_1 = \left\{ i \in I \mid a_i \neq 0_{A_i} \right\}$ and $I_2 = \left\{ i \in I \mid b_i \neq 0_{A_i} \right\}$, then I_1 and I_2 are finite sets and $a_i + b_i = 0_{A_i}$ for every $i \notin I_i \cup I_2$.

Therefore $\left\{ i \in I \mid a_i + b_i \neq 0_{A_i} \right\} \subset \left(I_1 \cup I_2 \right)$ which is finite.

This implies that $\{a_i\} + \{b_i\} = \{a_i + b_i\} \in \sum_{i\in I} A_i$.

Similarly, $\left\{ i \in I \mid r\,a_i \neq 0_{A_i} \right\} \subset I_1$ which is finite.

Therefore $r\{a_i\} = \{ra_i\} \in \sum_{i\in I} A_i$.

So $\sum_{i\in I} A_i$ is a submodule of $\prod_{i\in I} A_i$.

(iii) For $k \in I$, $\pi_k : \prod_{i\in I} A_i \to A_k$ is defined by $\pi_k\left(\{a_i\}\right) = a_k$.

Let $\{a_i\},\{b_i\} \in \prod_{i\in I} A_i$ and $r \in R$.

$$\begin{aligned}
\pi_k\left(\{a_i\} + \{b_i\}\right) &= \pi_k\left(\{a_i + b_i\}\right) \\
&= a_k + b_k \\
&= \pi_k\left(\{a_i\}\right) + \pi_k\left(\{b_i\}\right) \\
\pi_k\left(r\{a_i\}\right) &= \pi_k\left(\{r\,a_i\}\right) \\
&= ra_k \\
&= r\pi_k\left(\{a_i\}\right).
\end{aligned}$$

So π_k is an R-module homomorphism.

Now, we will show that π_k is surjective.

Take any element $x \in A_k$ and let $\{c_i\}$ be an element of $\prod_{i\in I} A_i$ such that $c_k = x$.

Then $\pi_k\left(\{c_i\}\right) = c_k = x$.

Therefore π_k is an R-module epimorphism.

(iv) For $k \in I$, $l_k : A_k \to \sum_{i\in I} A_i$ is defined by $l_k(x) = \{a_i\}$ where $a_k = x$ and $a_i = 0_{A_i}$ for $i \neq k$. Take any $x, y \in A_k$ and $r \in R$.

$$l_k(x+y) = \{b_i\} \qquad b_k = x + y \text{ and } b_i = 0_{A_i} \text{ for } i \neq k$$

$$l_k(x+y) = \{c_i\} + \{d_i\}$$

where $c_k = x$ and $c_i = 0_{A_i}$, $d_k = y$ and $d_i = 0_{A_i}$ for $i \neq k$

$$l_k(x+y) = l_k(x) + l_k(y)$$

$$l_k(rx) = \{c_i\} \text{ where } c_k = rx,\ c_i = 0_{A_i} \text{ for } i \neq k$$
$$= r\{f_i\} \text{ where } f_k = x,\ f_i = 0_{A_i} \text{ for } i \neq k$$
$$= rl_k(x).$$

So l_k is an R-module homomorphism.

Now, we will show that l_k is an injective.

$$l_k(x) = l_k(y) \Rightarrow \{c_i\} = \{d_i\} \text{ where } c_k = x \text{ and } c_i = 0_{A_i},\ d_k = y \text{ and } d_i = 0_{A_i} \text{ for } i \neq k$$
$$\Rightarrow c_k = d_k$$
$$\Rightarrow x = y$$

Therefore l_k is an R-module monomorphism. ■

Proposition.[2] Let $\{M_\alpha\}$ be a collection of left R-modules indexed by the set I, and let N be a left R-module. For each $\alpha \in I$, $P_\alpha : \prod_{\alpha\in I} M_\alpha \to M_\alpha$, P_α is a projection. Then for any set $\{f_\alpha\}_{\alpha\in I}$ of R-homomorphisms such that $f_\alpha : N \to M_\alpha$ for each $\alpha \in I$, there exists a unique R-homomorphism $f : N \to \prod_{\alpha\in I} M_\alpha$ such that $P_\alpha f = f_\alpha$ for all $\alpha \in I$.

Proof: Let $\{f_\alpha\}_{\alpha\in I}$ be any set of R-homomorphism such that $f_\alpha : N \to M_\alpha$ for each $\alpha \in I$.

Let $f : N \to \prod_{\alpha\in I} M_\alpha$.

If $x \in N$, then we define f(x) by letting its components $(f(x))_\alpha = f_\alpha(x)$ for each $\alpha \in I$.

But $P_\alpha(f(x)) = (f(x))_\alpha$

$P_\alpha(f(x)) = f_\alpha(x)$

$(P_\alpha f)(x) = f_\alpha(x),\ \forall x \in N$

Thus $P_\alpha f = f_\alpha$.

Hence f can be defined to satisfy $P_\alpha f = f_\alpha$ for all $\alpha \in I$.

Take any $x, y \in N$ and $r \in R$.

Thus $x + y \in N$ and $rx \in N$ since N is a left R-module.

Hence $[f(x+y)]_\alpha = f_\alpha(x+y)$

$= f_\alpha(x) + f_\alpha(y)$

$= [f(x)]_\alpha + [f(y)]_\alpha$.

For each $\alpha \in I$, the α^{th} component of $f(x+y)$ is the sum of the α^{th} component of f(x) and the α^{th} component of f(y).

$f(x+y) = f(x) + f(y)$

$[f(rx)]_\alpha = f_\alpha(rx)$

$= rf_\alpha(x)$

$= r[f(x)]_\alpha$.

For each α^{th} component of f(rx) is the product of r and the α^{th} component of f(x).

$f(rx) = rf(x)$.

Hence f is an R-homomorphism.

Let $g : N \to \prod_{\alpha\in I} M_\alpha$ be another R-homomorphism such that $p_\alpha g = f_\alpha$ for all $\alpha \in I$.

$$p_\alpha g = p_\alpha f \ , \ \forall \alpha \in I$$
$$(p_\alpha g)(x) = (p_\alpha f)(x) \ , \ \forall x \in N .$$
$$p_\alpha (g(x)) = p_\alpha (f(x))$$
$$[g(x)]_\alpha = [f(x)]_\alpha \ , \ \forall x \in N \text{ and } \forall \alpha \in I .$$
$$g(x) = f(x)$$
$$g = f$$

Thus f is a unique R-homomorphism. ■

Example. Let M be a left R-module. It can be shown that M is finitely generated if there exists a submodule $N \subseteq M$ such that N and M/N are both finitely generated.

Let M be a left R-module. Let $N \subseteq M$ be a submodule of M such that N and M/N are both finitely generated.

Therefore the elements $y_1, y_2, ..., y_n$ generate N and $\overline{x}_1, \overline{x}_2, ..., \overline{x}_k$ generate M/N where $\overline{x}_i = x_i + N$.

Take any $x \in M$.

In M/N,
$$\begin{aligned} \overline{x} &= x + N \\ &= \sum_{i=1}^{k} a_i \overline{x}_i \\ &= \sum_{i=1}^{k} a_i (x_i + N) \\ &= \sum_{i=1}^{k} ((a_i x_i) + N) \\ &= (\sum_{i=1}^{k} a_i x_i) + N . \end{aligned}$$

We get $\overline{x} = x + N = (\sum_{i=1}^{k} a_i x_i) + N$. So $(x - \sum_{i=1}^{k} a_i x_i) \in N$.

Since $y_1, y_2, ..., y_n$ generate N, $x - \sum_{i=1}^{k} a_i x_i = \sum_{j=1}^{n} b_j y_j$.

Therefore $x = \sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{n} b_j y_j$.

Then M is finitely generated.

Example. It can be proved that the projection mapping $p_\alpha : \prod_{\alpha\in I} M_\alpha \to M_\alpha$ defined by $p_\alpha(m) = m_\alpha$ is an R-homomorphism.

For each index $\alpha \in I$, the projection mapping $p_\alpha : \prod_{\alpha\in I} M_\alpha \to M_\alpha$ is defined by $p_\alpha(m) = m_\alpha$, $m \in \prod_{\alpha\in I} M_\alpha$.

$m = (m_1, m_2, ..., m_n, ...)$

$p_1(m) = m_1$, $p_2(m) = m_2$, …, $p_n(m) = m_n$, …

Take any $x, y \in \prod_{\alpha\in I} M_\alpha, r \in R$.

$$
\begin{aligned}
x = (x_\alpha) &= (x_1, x_2, ..., x_n, ...) \\
y = (y_\alpha) &= (y_1, y_2, ..., y_n, ...). \\
p_\alpha(x+y) &= (x+y)_\alpha \\
&= x_\alpha + y_\alpha = p_\alpha(x) + p_\alpha(y) \\
p_\alpha(rx) &= (rx)_\alpha \\
&= rp_\alpha(x)
\end{aligned}
$$

Thus p_α is an R-homomorphism.

Example. For each $\alpha \in I$ an inclusion mapping $i_\alpha : M_\alpha \to \prod_{\alpha\in I} M_\alpha$ is defined by $i_\alpha(x) = m$ where $m_\alpha = x$ and $m_\beta = 0$ for all $x \in M_\alpha$ and $\beta \neq \alpha$, it can be proved that i_α is an R-homomorphism. For each $\alpha \in I$ an inclusion mapping $i_\alpha : M_\alpha \to \prod_{\alpha\in I} M_\alpha$ is defined for all $x \in M_\alpha$ by $i_\alpha(x) = m$.

Take any $x, y \in M_\alpha$, $\alpha \in I$ and $r \in R$.

$x + y \in M_\alpha$ and $rx \in M_\alpha$ since M_α is a module.

$$
\begin{aligned}
i_\alpha(x) &= (0,0,...,x,0,...) \\
i_\alpha(y) &= (0,0,...,y,0,...) \\
i_\alpha(x+y) &= (0,0,...,x+y,0,...) \\
&= (0,0,...,x,0,...) + (0,0,...,y,0,...) \\
&= i_\alpha(x) + i_\alpha(y)
\end{aligned}
$$

Again,

$$
\begin{aligned}
i_\alpha(r\,x) &= (0,0,...,rx,0,...) \\
&= r(0,0,...,x,0,...) \\
&= r\, i_\alpha(x)
\end{aligned}
$$

Therefore i_α is an R-homomorphism.

RESULTS AND CONCLUSION

This theory is applied to obtain the structure of abelian group and the rational canonical and Jordan normal forms of matrices. The basic facts about rings and modules are given in full generality, so that some further topics can be discussed, including projective modules and the connection between modules and representations of groups. Furthermore, the results of this paper showed how the module structure of algebra plays a vital role in homomorphic signal processing in branch of engineering mainly in information technology, electronics and telecommunication engineering, computer science, etc.

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