Basic Ideas on Isomorphism of Graphs

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Abstract

In this paper, the homomorphism and isomorphism of two graphs with examples were studied. And some related properties of isomorphic graphs were also presented.

Keywords: preserve adjacency, invariant, permutation matrix

INTRODUCTION

The word isomorphism derives from Greek; iso means "equal", and morphosis means "to form" or "to shape". Graph isomorphism is a bijective function between two graphs; it preserves adjacency relation between the vertices of graphs.

OBJECTIVE

To get other forms of the given graph from the side of isomorphism.

Preliminaries

In this section, homomorphism and isomorphism of two graphs are defined, and automorphism of a graph and invariant are introduced.

Definition [3] A **homomorphism** from G to H is an ordered pair (f, g) of maps $f: V(G) \to V(H)$ and $g: E(G) \to E(H)$ satisfying the condition

$$(u, v) \in E(G) \text{ implies that } (f(u), f(v)) \in E(H).$$

That is, u and v are endvertices of edge e in G then f(u) and f(v) are corresponding endvertices of the mapped edge in H.

Example. Consider the two graphs G and H in Figure 1. We define the map $f: V(G) \to V(H)$ as $f(a) = f(b) = f(d) = 1$, $f(c) = 2$ and $f(e) = 3$. Also we define the map $g: E(G) \to E(H)$ as $g(e_1) = g(e_2) = g(e_6) = x_1$, $g(e_3) = g(e_4) = g(e_5) = x_2$, $g(e_7) = x_3$.

Figure 1

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We can check for each edge $e_1, e_2, ..., e_7$ of G by

$$e_1 = (a,a), g(a,a) = g(e_1) = x_1 = (1,1) = (f(a), f(a))$$
$$e_2 = (a,b), g(a,b) = g(e_2) = x_1 = (1,1) = (f(a), f(b))$$
$$e_3 = (b,c), g(b,c) = g(e_3) = x_2 = (1,2) = (f(b), f(c))$$
$$e_4 = (b,c), g(b,c) = g(e_4) = x_2 = (1,2) = (f(b), f(c))$$
$$e_5 = (c,d), g(c,d) = g(e_5) = x_2 = (2,1) = (f(c), f(d))$$
$$e_6 = (a,d), g(a,d) = g(e_6) = x_1 = (1,1) = (f(a), f(d))$$
$$e_7 = (c,e), g(c,e) = g(e_7) = x_3 = (2,3) = (f(c), f(e)).$$

Hence (f, g) is a homomorphism.

Figure 2

Also we define the maps f and g for the Figure 2 as follows, $f: V(G) \to V(H)$ by $f(a) = f(c) = 1$, $f(b) = f(d) = 2$, $f(e) = 3$ and define the map $g: E(G) \to E(H)$ by $g(e_1) = g(e_2) = g(e_3) = x_1$, $g(e_4) = g(e_6) = x_2$, $g(e_5) = g(e_7) = x_3$. We can check for each edge $e_1, e_2, ..., e_7$ of G by

$$e_1 = (a,b), g(a,b) = g(e_1) = x_1 = (1,2) = (f(a), f(b))$$
$$e_2 = (b,c), g(b,c) = g(e_2) = x_1 = (2,1) = (f(b), f(c))$$
$$e_3 = (c,d), g(c,d) = g(e_3) = x_1 = (1,2) = (f(c), f(d))$$
$$e_4 = (d,e), g(d,e) = g(e_4) = x_2 = (2,3) = (f(d), f(e))$$
$$e_5 = (a,e), g(a,e) = g(e_5) = x_3 = (1,3) = (f(a), f(e))$$
$$e_6 = (b,e), g(b,e) = g(e_6) = x_2 = (2,3) = (f(b), f(e))$$
$$e_7 = (c,e), g(c,e) = g(e_7) = x_3 = (1,3) = (f(c), f(e)).$$

Hence (f, g) is a homomorphism.

Definition [3] Two graphs G and H are **isomorphic** if there is a one-to-one, onto function f from the vertices of G to the vertices of H and a one-to-one, onto function g from the edges of G to the edges of H, so that an edge e is incident on v and w in G if and only if the edge g(e) is incident on f(v) and f(w) in H. The pair of functions f and g is called an **isomorphism** of G and H. That is, a homomorphism from G to H is an ordered pair of maps (f, g) satisfying the compatibility condition $g(u,v) = (f(u), f(v))$. An isomorphism from G to itself is called an **automorphism** of G.

Example. Consider the two graphs G and H as shown in Figure 3.

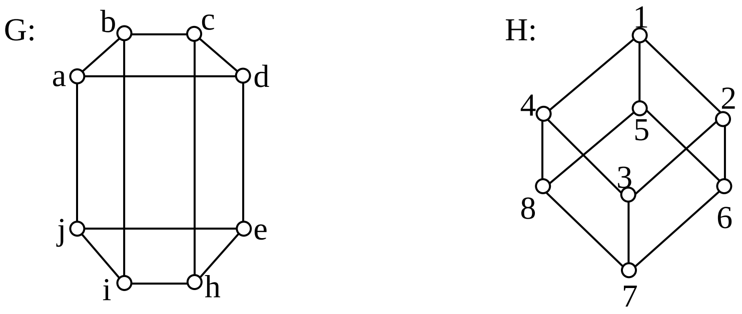

Figure 3

The function $f: V(G) \to V(H)$ is defined by $f(a)=1, f(b)=2, f(c)=3, f(d)=4$, $f(e)=8, f(h)=7, f(i)=6, f(j)=5$. Also define $g((x,y))=(f(x),f(y))$ for every $(x,y)\in E(G)$. Then f and g are one-to-one correspondence. Thus (f, g) is an isomorphism of G onto H.

Hence $G \cong H$.

Example. Consider the following Figure 4. The function $f: V(G) \to V(H)$ is defined by $f(a)=1, f(b)=5, f(c)=3, f(d)=6, f(e)=4, f(h)=2$.

However $g((5,3)) \neq (f(b),f(c))$ for $(b,c) \notin E(G)$. Hence these graphs are not isomorphic.

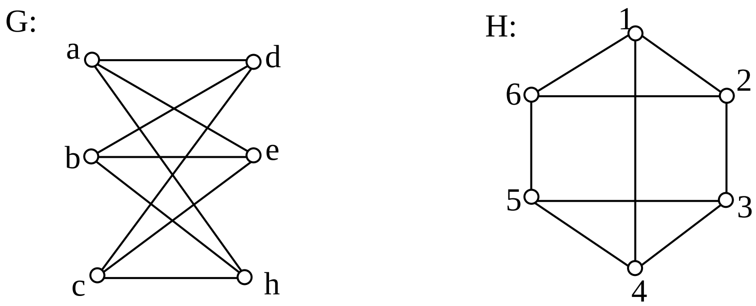

Figure 4

Definition [1] If the graph H does not have a property of the graph G, that property is called an invariant. But if the graphs G and H were isomorphic, the properties of two graphs are the same.

Example. Consider the following Figure 5.

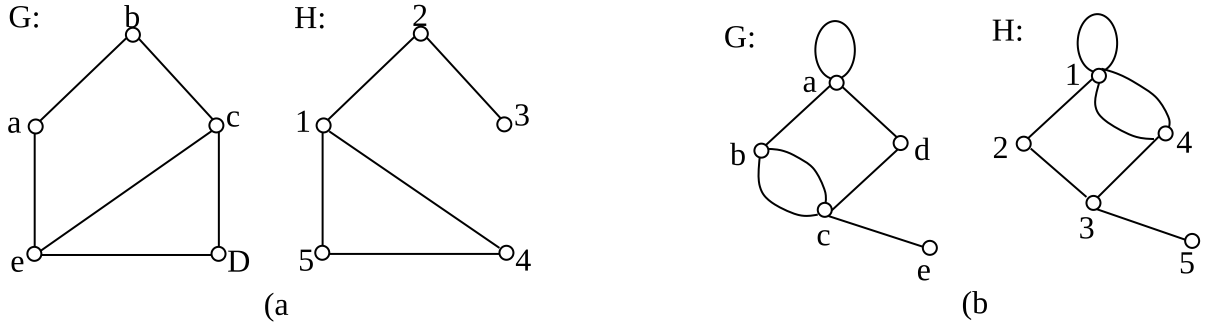

Figure 5

For Figure 5(a), the invariant is that the graph G has six edges but H does not. In Figure 5(b), the graphs G and H have the same number of vertices, edges, a pair of parallel edges and a loop. The invariant is a vertex 1of degree five in H but not in G.

Some Properties of Isomorphic Graphs

In this section we discuss some properties of isomorphic graphs and also we study more on isomorphisms on simple graphs.

Theorem [3] Graphs G and H are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal.

Proof: See [3]. □

Corollary [3] Let G and H be simple graphs. The following are equivalent: (i) The graphs G and H are isomorphic. (ii) There is a one-to-one, onto function f from the vertex set of G to the vertex set of H satisfying the following: Vertices u and v are adjacent in G if and only if the vertices f(u) and f(v) are adjacent in H.

Proof: See [3]. □

Theorem [2] The isomorphism relation between graphs in a fixed collection of graphs is an equivalence relation. That is, (i) For any graph G, $G \cong G$. (ii) If $G \cong G'$ and $G' \cong G''$ then $G \cong G''$. (iii) If $G \cong G'$ then $G' \cong G$.

Proof: (i) The function $f: V(G) \to V(G)$ is the map defined by $f(v_i) = v_i$ for each $i \in N$, then $g: E(G) \to E(G)$ must satisfy $g((u,v)) = (f(u), f(v))$ for every $(u,v) \in E(G)$. Therefore f and g are bijective maps. Hence $G \cong G$.

(ii) Let $\phi = (f_1, g_1)$ be an isomorphism of G onto G' and $\psi = (f_2, g_2)$ be an isomorphism of G' onto G''. We have to show that $\psi \circ \phi = (f_2 \circ f_1, g_2 \circ g_1)$ is an isomorphism of G onto G''. Since ϕ is bijective, $g_1((u,v)) = (f_1(u), f_1(v))$ for every $(u,v) \in E(G)$.

Also ψ is bijective, $g_2((f_1(u), f_1(v))) = (f_2(f_1(u)), f_2(f_1(v)))$, for every $(f_1(u), f_1(v)) \in G'$.

Thus $g_2(g_1(u,v)) = (f_2((f_1(u)), f_2((f_1(v)))$ for every $g_1(u,v) \in E(G')$

$$(g_2 \circ g_1)(u,v) = ((f_2 \circ f_1)(u), (f_2 \circ f_1)(v)) \text{ for every } (u,v) \in E(G).$$

Therefore $\psi \circ \phi = (f_2 \circ f_1, g_2 \circ g_1)$ is an isomorphism of G onto G''. Hence $G \cong G''$.

(iii) Let (f,g) be an isomorphism of G onto G'. Assume that $e = (u,v)$ and $g^{-1}(e) = (x,y)$. We have to show that $(x,y) = (f^{-1}(u), f^{-1}(v))$. Since g is onto $g^{-1}(e) = e_1$, so $g(e_1) = e$ and $e_1 = (x,y)$. Since (f,g) is an isomorphism, $(u,v) = g((x,y)) = (f(x), f(y))$ and also $f^{-1}(u) = x$, $f^{-1}(v) = y$. Therefore $(f^{-1}(u), f^{-1}(v)) = (x,y)$. This proves that (f^{-1}, g^{-1}) is an isomorphism.

Hence $G' \cong G$. □

Theorem [3] For simple graphs G and H, G and H are isomorphic if and only if $\overline{G}$ and $\overline{H}$ are isomorphic where $\overline{G}$ and $\overline{H}$ are the complement of graphs G and H.

Proof: Suppose that G and H are isomorphic. Then the pair of the function (f, g) is one-to-one correspondence between G and H. We can also construct this isomorphism for $\overline{G}$ and $\overline{H}$. The function f is unchanged. Let (u, v) be an edge in $\overline{G}$. Set $g(u,v) = (f(u), f(v))$. It can be

□

verified that the function f and g provide an isomorphism of $\overline{G}$ and $\overline{H}$. If $\overline{G}$ and $\overline{H}$ are isomorphic, by the preceding result, $\overline{\overline{G}} = G$ and $\overline{\overline{H}} = H$ are isomorphic.

Definition [1] Two $n \times n$ matrices X and Y are **orthogonally equivalent** if there is a permutation matrix P such that $Y = P^T X P$.

Remark [1] The following statements are equivalent: (i) The graphs G and H are isomorphic. (ii) The adjacency matrices A_G and A_H are orthogonally equivalent with respect to any labeling of their vertices.

Example. Consider two graphs G and H as shown in Figure 6. The function $f : V(G) \rightarrow V(H)$ is defined by f(1) = **3**, f(2) = **5**, f(3) = **1**, f(4) = **2**, and f(5) = **4**. Also we have the $g((x, y)) = (f(x), f(y))$ for every $(x, y) \in E(G)$.

Hence $G \cong H$.

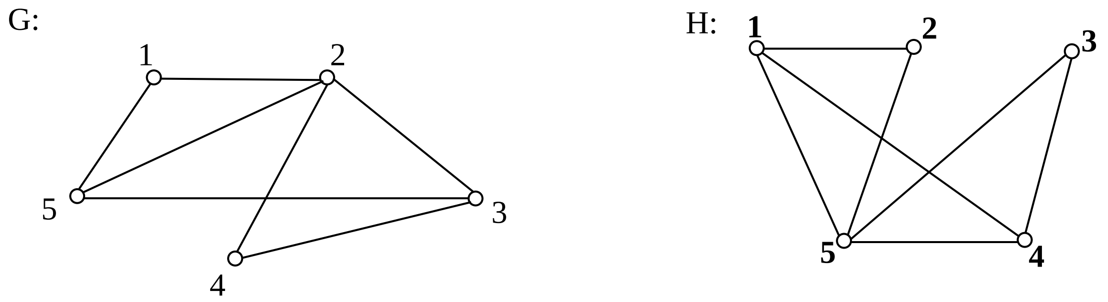

Figure 6

$$A_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}, \quad A_H = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}, \quad P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$P^T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$ We get $P^T A_H P = A_G$.

RESULTS AND DISCUSSIONS

(1) If two graphs are isomorphic, they must have the same number of components.

(2) Two graphs G and H are isomorphic if and only if the corresponding subgraph of G and H obtained by deleting some vertices in G and other images in H are isomorphic.

(3) To check whether the graphs G and H on n vertices are isomorphic, it suffices to consider that there exists possible homomorphisms from G into H induced by the $n!$ bijective vertex mappings from V(G) onto V(H).

(4) To check whether two simple graphs G and H are isomorphic in which G and H have maximum degree of k and with n_i vertices of degree i for $1 \leq i \leq k$ where $n = n_0 + n_1 + \cdots + n_k$, it suffices to consider the possible homomorphisms from G into H induced by at most $n_1!n_2!\cdots n_k!$ bijective vertex mapping of V(G) onto V(H).

(5) Two graphs G and H are isomorphic then they must have the same number of h-cliques, for each $h \in N$.

(6) Two graphs G and H on n vertices and m edges are isomorphic if, and only if, there are $n \times n$ and $m \times m$ permutation matrices P and Q, respectively, such that $B_H = P^T B_G Q$, where B_G and B_H are incidence matrices of G and H.

Non-Isomorphic Graphs

There is only one non-isomorphic graph of order 1, two non-isomorphic graphs of order 2, four non-isomorphic graphs of order 3, and eleven non-isomorphic graphs of order 4. These can be seen in **[1]**. Now thirty four non-isomorphic graphs of order 5 are illustrated in Figure 7 below.

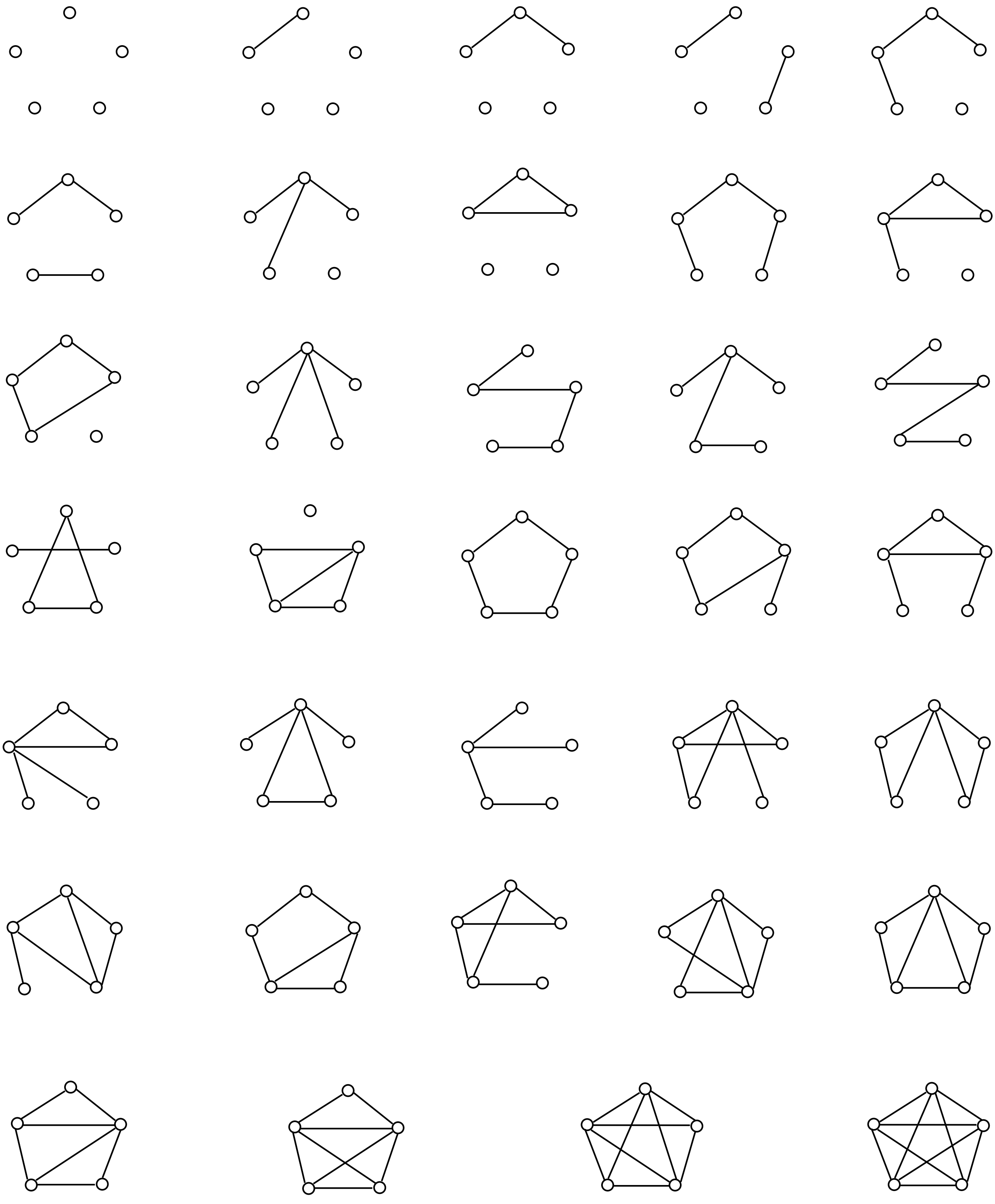

Figure 7

We state the number of non-isomorphic graphs of the given graph with n vertices in Table 1 where $n = 1, 2, ..., 9$.[4]

Number of Vertices (n)	1	2	3	4	5	6	7	8	9
Graphs $2^{\binom{n}{2}}$	1	2	8	64	1024	32768	2097152	268435456	68719476736
Non-isomorphic Graph	1	2	4	11	34	156	1044	12346	274668

Table 1

APPLICATIONS

Isomorphism phenomena is used to reason about the behavior of complex systems and to check many things, computer programs and logic proof in parallel processing. To store the circuit of the database in the form of a colored graph and the matching user input circuit with it using graph isomorphism. Isomorphism idea is mainly used for showing that two languages are equal. Also isomorphism concept is used in image processing, protein structure, computer and information system, chemical bond structure and social networks. To classify the graph classes, isomorphism testing is used.

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