Some Concepts of Posets

Swe Swe Yu[1], Zon Zon Naing[2], May Myat Thu[3]

Abstract

In this paper, we present the definitions and background information about posets. Then we construct the Hasse diagram for the posets and describe the indicated sets and indicated relations of posets with Hasse diagram. Next, we discuss the notions of isomorphic, dual and self dual of posets. Moreover, we study order ideal of posets, and we will see how these objects relate to posets.

Keywords: Posets, Hasse diagram, Isomorphic of posets, Dual of posets, Order ideal of posets.

INTRODUCTION

This paper is organized as follows: In first part, we present basic definitions and examples of posets. And then we describe some basic properties and results on the posets. In second part, we discuss the complete description of this poset is given for isomorphic of posets. In third part, we study the notions of dual of posets and some results. In fourth part, we consider the order ideal of posets and some related fact. In the final part, we indicate some conclusions and further research directions.

Posets and Simple Results

In this part, we present the basic definitions and examples of posets. And then we discuss the properties of posets.

Definitions.[1] A non empty set *P*, together with a binary relation R is said to form a ***partially ordered set*** or a ***poset*** if the following conditions hold:

P*1 :** ***Reflexivity : aRa for all $a \in P$

P*2 :** ***Anti-Symmetry : If $aRb, bRa,$ then $a = b$ $(a,\ b \in P)$

P*3 :** ***Transitivity : If $aRb,\ bRc,$ then aRc $(a,\ b,\ c \in P)$

It is denoted by (P, R). We generally use the symbol $\le$ in place of *R*. We write $a < b$ if $a \le b$ and $a \ne b$.

Examples (i) The set $\mathbb{N}$ of natural numbers forms a poset under the usual $\le$. Similarly, the integers, rationals and real numbers also form posets under usual $\le$.

(ii) The set $\mathbb{N}$ of natural numbers under divisibility forms a poset. Thus here, $a \le b$ means $a \mid b$ (*a* divides *b*).

(iii) Let $\mathcal{F}$ be a collection of sets $A, B, C, \ldots$. Then $\mathcal{F}$ under "contained in" $\subseteq$ relation forms a poset.

[1] Professor, Dr, Department of Mathematics, Hinthada University
[2] Tutor, Department of Mathematics, Hinthada University
[3] Tutor, Department of Mathematics, Hinthada University

Definitions.[2] If X is a poset, $Y \subseteq X$ is a ***chain*** or a ***totally ordered set*** or a ***toset*** if for all $y_1, y_2 \in Y$ either $y_1 \le y_2$ or $y_2 \le y_1$ (in other words if Y is a poset in which every two members are ***comparable***). Y is an ***antichain*** if any two distinct elements of Y are incomparable.

Remark. Two elements of a poset may not be comparable. For instance, 2 and 3 are not comparable in Example (ii) above although these are comparable in Example (i). The posets in Example (i) are all chains whereas those in Example (ii) and (iii) are not chains. Clearly also if x, y are distinct elements of a chain then either $x < y$ or $y < x$.

Proposition.[1] A non empty subset S of a poset P is a poset and if P is a chain then so is S (under the same relation, restricted to S).

Proof. For $a, b, c \in S$ implies $a, b, c \in P$. Thus $a \le a$ for all a. Moreover, $a \le b,\ b \le a$ imply $a = b$ and $a \le b,\ b \le c$ imply $a \le c$. Therefore, S is a poset. Again if P is a chain then S would also be a chain ($a, b \in S$ implies $a, b \in P$ so a, b are comparable).

Proposition.[1] In a poset $a < a$ for no a and $a < b$, $b < c$ imply $a < c.$

Proof. Suppose there exists some element a in a poset P such that $a < a.$ Then by definition, $a \le a$ and $a \ne a.$ By anti-symmetry $a \le a$, $a \le a$ imply $a = a.$ Thus, we get a contradiction.

Again, $a < b$, $b < c$ imply $a \le b,\ a \ne b,$

$$b \le c,\ b \ne c.$$

So $a \le b$, $b \le c$ imply $a \le c$ (by transitivity).

If $a = c,$ then $c \le b$, $b \le c$ imply $b = c,$ a contradiction. Hence $a < c.$

Definition.[3] For $x, y \in P,$ when $x < y$ and there does not exist $z \in P$ such that $x < z < y,$ we say that y ***covers*** x.

Example (i). Consider the set $\{1,\ 2,\ 3\}$. Let $B_3 = \mathcal{P}(\{1, 2, 3\})$ denote the power set of $\{1,\ 2,\ 3\}$, that is, all subsets of $\{1,\ 2,\ 3\}$, with the partial ordering given by set inclusion, $\subseteq$.

Thus $B_3 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,\ 2\}, \{1,\ 3\}, \{2,\ 3\}, \{1,\ 2,\ 3\}\}.$

Let's carefully go through the definition to see that $\subseteq$ in B_3 satisfies all the properties of a partial order relation. We rely on standard results from set theory.

Reflexivity: For any element $S \in B_3$, it follows from the definition of $\subseteq$ that $S \subseteq S,$ since every set has itself as a subset.

Anti-symmetry: Let $S, T \in B_3$. If $S \subseteq T$ and $T \subseteq S,$ then by definition of set equality $S = T.$

Transitivity: Let $R, S, T \in B_3,$ suppose $R \subseteq S$ and $S \subseteq T.$ It is easy to show that this implies $R \subseteq T.$ Thus B_3 is a poset under set inclusion. In fact, any collection of subsets forms a poset under $\subseteq$. We can generalize B_3 to B_n.

Now, consider $\{1\}$ and $\{1,\ 3\}$. Since $\{1\} \subseteq \{1,\ 3\}$, these two elements are comparable. Next, consider $\{1,\ 3\}$ and $\{2,\ 3\}$, neither of these elements are subsets of the other. Hence, they are incomparable. We consider $\{1,\ 2\}$ covers $\{1\}$, but although $\{1,\ 2,\ 3\} \supseteq \{1\}$, $\{1,\ 2,\ 3\}$ does not cover $\{1\}$, since we have $\{1\} \subseteq \{1,\ 2\} \subseteq \{1,\ 2,\ 3\}$.

We can represent a poset as a directed graph with elements of the poset as nodes, where $x, y \in P$ have an edge between them in the graph if x covers $y \in P$. This graph is called the ***Hasse diagram*** of *P*. The Hasse diagram of B_3 is given in Figure 1.

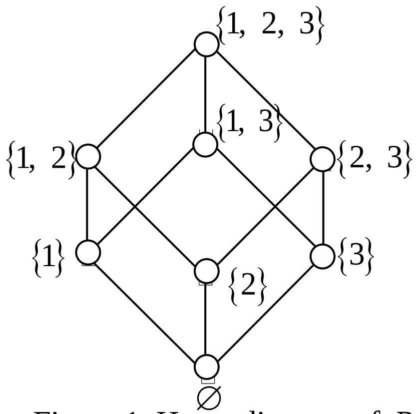

Figure 1. Hasse diagram of B_3

As we can see in Figure 1, instead of using edges with arrows, the Hasse diagram is drawn such that if x covers $y \in P$ then x is drawn above y in the diagram, with an edge between them.

(ii) The integers are a poset with the usual ordering $\leq$. This poset's Hasse diagram is an infinite line of elements, as seen in Figure 2.

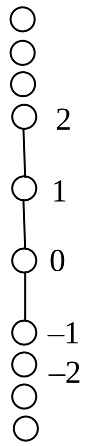

Figure 2. Hasse diagram for the integers

Definition.[1] Let *P* be a poset. If there exists an element $a \in P$ such that $x \leq a$ for all $x \in P$ then *a* is called ***greatest*** or ***unity element*** of *P*. It is generally denoted by *u*.

Proposition.[1] The greatest element of poset *P* (if it exists) is unique.

Proof. Let *x* and *y* be greatest elements of poset *P*. By definition of greatest, we must have $y \leq x$ as *x* is the greatest element of poset *P* and *y* belongs to poset *P*. Likewise $x \leq y.$ By anti-symmetry $x = y.$ Thus, any greatest element is unique.

Definition.[1] Let *P* be a poset. If there exists an element $b \in P$ such that $b \leq x$ for all $x \in P$ then *b* is called ***least*** or ***zero element*** of *P*. It is generally denoted by 0. If a poset *P* has least and greatest elements, we call it a ***bounded poset***. Indeed $0 \leq x \leq u$ for all $x \in P.$

Proposition.[1] The least element of poset *P* (if it exists) is unique.

Proof. Let *x* and *y* be least elements of poset *P*. By definition of least, we must have $x \le y$ as *x* is the least element of poset *P* and *y* belongs to poset *P*. Likewise $y \le x.$ By anti-symmetry $x = y.$ Thus, any least element is unique.

Example. Let $X = \{1, 2, 3\}$. Then ($\mathcal{P}$ (*X*), $\subseteq$) is a poset. Let $A = \{\varnothing, \{1, 2\}, \{2\}, \{3\}\}$. Then $(A, \subseteq)$ is a poset with $\varnothing$ as least element. *A* has no greatest element. Let $B = \{\{1, 2\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. Then *B* has greatest element $\{1, 2, 3\}$ but no least element. If $C = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ then *C* has both least and greatest elements namely, $\varnothing$ and $\{1, 2\}$. Thus *C* is bounded poset. Again $D = \{\{1\}, \{2\}, \{1, 3\}\}$ has neither least nor greatest element.

Definition.[1] An element *a* in a poset *P* is called ***maximal element*** of *P* if $a < x$ for no $x \in P.$

Remarks.

(i) A poset may not have a maximal element. For instance, the natural numbers under usual $\le$ have no maximal element.

(ii) A poset may have more than one maximal element. In the poset $\{2, 3, 4, 6\}$ under divisibility 4 and 6 are both maximal elements (none being the greatest).

(iii) Maximal element may not be the greatest element as seen in (ii) above.

(iv) Greatest element is the unique maximal element of a poset *P*. Indeed, suppose *a* is the greatest element of *P*. Then $x \le a \ \ \forall x \in P.$ If *a* is not maximal, then there exists some $y \in P$ such that $a < y.$ That is, $a \le y, \ a \ne y.$ But by the definition of greatest, we get $y \le a$ and so $y = a,$ a contradiction. Hence, *a* is maximal. Again, suppose *b* is another maximal element of *P*. Since *a* is greatest and $b \in P, \ b \le a.$ But $b \ne a$ and so $b < a,$ a condraction as *b* is maximal. Hence, greatest element is unique maximal element of a poset *P*.

Definition.[1] An element *b* in a poset *P* is called a ***minimal element*** of *P* if $x < b$ for no *x* in *P*.

Remark. We can state and prove similar results for minimal elements as done for maximal elements.

Theorem.[1] If *S* is a non empty finite subset of a poset *P*, then *S* has maximal and minimal elements.

Proof. Let $x_1, x_2, \ldots, x_n$ be all the distinct elements of *S* in any random order. If x_1 is maximal element, we are done. If x_1 is not maximal, then there exists some $x_i \in S$ such that $x_1 < x_i$. If x_i is maximal, we are done. If not, there exists some $x_j \in S$ such that $x_i < x_j$. Continuing like this, we'll reach a stage where some element will be maximal. Similarly, we can show that *S* has minimal elements.

Isomorphic of Posets

In this part, we discuss about isomorphic of posets, which are the main objects of study in the next part.

Definitions.[1] Let (P, R) and (Q, R') be two posets. A one-one onto map $f : P \to Q$ is called an ***isomorphism*** if $x R y \Leftrightarrow f(x) R' f(y),\ x, y \in P.$ We say that P and Q are called ***isomorphic*** of posets. We write, in that case, $P \cong Q.$ A mapping $f : P \to Q$ is called ***isotone*** or ***order-preserving*** if $x \le y \Rightarrow f(x) \le f(y).$

Theorem.[1] A mapping $f : P \to Q$ is an isomorphism if and only if f is isotone and has an isotone inverse.

Proof. Let $f : P \to Q$ be an isomorphism. Then f being one-one, onto, f^{-1} exists and is one-one, onto. Again, by definition of isomorphism, f will be isotone. We show $f^{-1} : Q \to P$ is also isotone. Let $y_1, y_2 \in Q$ where $y_1 \le y_2.$ Since f is onto, there exists $x_1, x_2 \in P$ such that $f(x_1) = y_1,\ f(x_2) = y_2 \Leftrightarrow x_1 = f^{-1}(y_1),\ x_2 = f^{-1}(y_2).$

Now $y_1 \le y_2 \Rightarrow f(x_1) \le f(x_2)$

$$\Rightarrow x_1 \le x_2$$

$$\Rightarrow f^{-1}(y_1) \le f^{-1}(y_2)$$

$$\Rightarrow f^{-1} \text{ is isotone.}$$

Conversely, let f be isotone such that f^{-1} is also isotone. Since f^{-1} exists, f is one-one, onto. Again, as f is isotone $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2),\quad x_1, x_2 \in P.$

Also f^{-1} is isotone implies

$$f(x_1) \le f(x_2) \Rightarrow f^{-1}(f(x_1)) \le f^{-1}(f(x_2))$$

$$\Rightarrow x_1 \le x_2$$

thus $$x_1 \le x_2 \Leftrightarrow f(x_1) \le f(x_2).$$

Hence f is an isomorphism.

Dual of Posets

In this part, we now present the concept of dual of posets.

Definition.[1] Let ρ be a relation defined on a set X. Then ***converse*** of ρ (denoted by $\bar{\rho}$) is defined by $a\bar{\rho}b \Leftrightarrow b\rho a,\ \ a, b \in X.$

Theorem.[1] If a set X forms a poset under a relation $\rho,$ then X forms a poset under $\bar{\rho},$ the converse of $\rho.$

Proof. $a\bar{\rho}a$ as $a\rho a$ for all $a \in X$ shows $\bar{\rho}$ is reflexive. Let $a\bar{\rho}b$ and $b\bar{\rho}a.$ Then $b\rho a$ and $a\rho b$ i.e., $a\rho b$ and $b\rho a$ imply $a = b.$ Thus $\bar{\rho}$ is anti-symmetric. Let $a\bar{\rho}b$ and $b\bar{\rho}c.$ Then $b\rho a,\ c\rho b$ or $c\rho b,\ b\rho a.$ So that $c\rho a$ implies $a\bar{\rho}c$ or that $\bar{\rho}$ is transitive and hence is a partial ordering. Thus, X forms a poset under a relation $\bar{\rho}.$

Definition.[1] If (X, ρ) be a poset, then the poset $(\bar{X}, \bar{\rho})$, where $\bar{X} = X$ and $\bar{\rho}$ is converse of ρ is called ***dual*** of *X*.

Theorem.[1] If (X, ρ) be a poset, then $X \cong \bar{\bar{X}}$, where $\bar{\bar{X}}$ is dual of $\bar{X}$.

Proof. Define $f : X \to \bar{\bar{X}}$ such that $f(x) = x,\ x \in X$, f is then clearly a well defined one-one onto map. Again, $x\rho y \Leftrightarrow y\bar{\rho}x \Leftrightarrow x\bar{\bar{\rho}}y \Leftrightarrow f(x)\bar{\bar{\rho}}f(y)$. Shows *f* is an isomorphism. ($\bar{\bar{\rho}}$ being converse of $\bar{\rho}$). Thus $X \cong \bar{\bar{X}}$.

Definition.[1] If a poset *X* is isomorphic to its dual $\bar{X}$, then *X* is called ***self dual***.

Example. Let $X \neq \varnothing$. Then the poset $(\mathcal{P}(X), \subseteq)$ of all subsets of *X* is self dual as we can define $f: \mathcal{P}(X) \to \overline{\mathcal{P}(X)}$ such that $f(A) = X - A$. If $f(A) = f(B)$ implies $X - A = X - B$ so $A = B$ or that *f* is one-one. Ontoness of *f* is obvious. Again $A \subseteq B$ in $\mathcal{P}(X) \Leftrightarrow X - A \supseteq X - B$ $\Leftrightarrow f(A) \supseteq f(B)$ in $\overline{\mathcal{P}(X)}$. Thus *f* is an isomorphism and hence $\mathcal{P}(X) \cong \overline{\mathcal{P}(X)}$. Therefore, the poset $(\mathcal{P}(X), \subseteq)$ is self dual.

Definition.[1] Let *A* and *B* be two posets. Then we can show that $A \times B = \{(a,b) \mid a \in A,\ b \in B\}$ forms a poset under the relation defined by $(a_1,b_1) \leq (a_2,b_2) \Leftrightarrow a_1 \leq a_2$ in $A, b_1 \leq b_2$ in *B*. It is clear that the three relations $\leq$ occurring above are different, being the respective relations in $A \times B$, A and *B*.

Reflexivity: $(a,b) \leq (a,b)$ for all $(a,b) \in A \times B$ as $a \leq a$ in A and $b \leq b$ in B for all $a \in A$, $b \in B$.

Anti-symmetry: Let $(a_1,b_1) \leq (a_2,b_2)$ and $(a_2,b_2) \leq (a_1,b_1)$. Then $a_1 \leq a_2,\ b_1 \leq b_2$ and so $a_1 = a_2,\ b_1 = b_2$. Thus $(a_1,b_1) = (a_2,b_2)$.

Transitivity: Let $(a_1,b_1) \leq (a_2,b_2)$ and $(a_2,b_2) \leq (a_3,b_3)$. Then $a_1 \leq a_2,\ b_1 \leq b_2$ and $a_2 \leq a_3,\ b_2 \leq b_3$ and so $a_1 \leq a_3$ and $b_1 \leq b_3$. Thus $(a_1,b_1) \leq (a_3,b_3)$.

We thus conclude that product of two posets is a poset. It is also called the ***direct*** or ***cardinal product*** of posets.

Theorem.[1] The cardinal product of two self dual posets is self dual.

Proof. Let *A* and *B* be the given self dual posets. Let $f : A \to \bar{A}$ and $g : B \to \bar{B}$ be the isomorphisms. Define $h : A \times B \to \overline{A \times B}$ such that $h((a,b)) = (f(a), g(b))$ then h is well defined, one-one map as

$$
\begin{aligned}
(a_1,b_1) = (a_2,b_2) &\Leftrightarrow a_1 = a_2, b_1 = b_2 \\
&\Leftrightarrow f(a_1) = f(a_2),\ \mathrm{g}(b_1) = g(b_2) \\
&\Leftrightarrow (f(a_1), \mathrm{g}(b_1)) = (f(a_2), g(b_2)) \\
&\Leftrightarrow h((a_1,b_1)) = h((a_2,b_2)).
\end{aligned}
$$

Again, for any $(x, y) \in \overline{A \times B}$, since and f^{-1}, g^{-1} exists as *f*, *g* are one-one onto, thus we get *h* is onto.

Next, $(a_1,b_1)\le(a_2,b_2)\Leftrightarrow a_1\le a_2, b_1\le b_2$

$$\Leftrightarrow f(a_1)\ge f(a_2),\ \ g(b_1)\ge g(b_2)$$

$$\Leftrightarrow (f(a_1),g(b_1))\ge(f(a_2),g(b_2))$$

$$\Leftrightarrow h((a_1,b_1))\ge h((a_2,b_2)).$$

Thus, *h* is the required isomorphism and $A\times B\cong\overline{A\times B}$. Therefore, the cardinal product of two self dual posets is self dual.

ORDER IDEAL OF POSETS

In this part, we introduce the notions of order ideal of posets.

Definition.[3] An ***order ideal*** of *P* is a subset *I* of *P* such that if $x\in I$ and $y\le x$ in *P*, then $y\in I$. This means that we can choose a subset of *P*, and form an order ideal consisting of that subset, as well as everything "below" those elements in *P*.

Examples. (i) Consider the poset *P* in Figure 3. For any subset *S* of the elements of *P*, we can form an order ideal $I(S)$ by taking *S* and everything below it. Some examples of order ideals are $I(\{2,\ 4\})=\{4,\ 3,\ 2,\ 1\}$ and $I(\{6,\ 1\})=\{6,\ 4,\ 3,\ 1\}$.

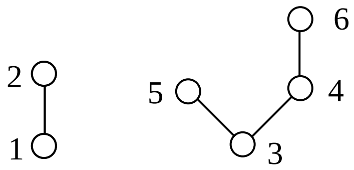

Figure 3. A Poset *P*

However, $J=\{5,\ 2,\ 1\}$ is not an order ideal because $3\le 5$ in *P*, but $3\notin J$. But $J'=\{5,\ 3,\ 2,\ 1\}=I(\{2,\ 5\})$ is an order ideal. The set of order ideals of *P* is {∅,{1}, {3}, {1, 3}, {1, 2}, {3, 5}, {3, 4}, {1, 2, 3}, {1, 3, 5}, {1, 3, 4}, {3, 4, 6}, {3, 4, 5}, {1, 2, 3, 4}, {1, 2, 3, 5}, {1, 3, 4, 5 }, {1, 3, 4, 6}, {3, 4, 5, 6}, {1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 6}

(ii) It is a fact that for any given order ideal *I*, there exists an antichain *S* such that $I=I(S)$. $I=\{1,\ 2,\ 3,\ 4,\ 5\}=I(\{2,\ 4,\ 5\})$, where $\{2,\ 4,\ 5\}$ is an antichain in *P*.

Definition.[3] The set of all order ideals of *P*, ordered by inclusion, forms a poset denoted $\boldsymbol{J(P)}$.

Example. Consider the poset $P=\{1,\ 2,\ 3\}$, with $2<3$, as seen in Figure 4. The set of order ideals ordered by inclusion gives us the poset $J(P)=\{\varnothing,\{1\},\{2\},\{1,\ 2\},\{2,\ 3\},\{1,\ 2,\ 3\}\}$.

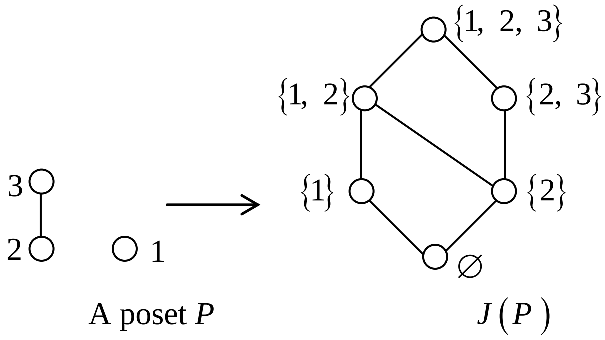

Figure 4. A poset P and its corresponding poset of order ideals, $J(P)$

RESULTS AND CONCLUSIONS

According to results of the present study, it is easy to find the set of minimal elements and maximal elements of a poset. In a general poset there may be no maximal element, or there may be more than one. But in a finite poset there is always at least one maximal element and minimal element. And then it is found that the greatest element and least element of a poset are unique. Using the notion of isomorphism, we discuss the relation between the isomorphism, isotone and isotone inverse. By the concepts of duality, we study the dual of dual poset is a poset and the product of two posets is a poset. It is also called the ***direct*** or ***cardinal product*** of posets. The definition can be extended to product of more than two posets in a similar way. Moreover, the cardinal product of two self dual posets is self dual. At the end of this paper, we find a poset and its corresponding poset of order ideal. Partially ordered sets arise very frequently in every life. Often it becomes natural and necessary to compute a ranking of the posets that respects the partial order's comparability. Moreover, we will explore a particular type of poset known as a ***lattice***. Lattices have a number of applications, and they provide one way for us to introduce and become familiar with ***Boolean Algebra***, a field of prime importance to computer science.

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