Prediction of Exponential lifetime by using Bayesian Method

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Abstract

Consider the problem of predictive intervals for future observations from an exponential distribution in two cases (i) fixed sample size (*FSS*) and (ii) random sample size (*RSS*). Next derive the predictive function for both *FSS* and *RSS* in closed forms. And then the upper and lower 1%, 2.5%, 5% and 10% critical points for the predictive functions are calculated. To show usefullness of result that present with some examples.

Keywords: Bayesian prediction, order statistics, predictive function,predictive intervals, random sample size.

Introduction

In many applications, the sample size could be random rather than fixed, the prediction problem for future samples based on *FSS* and *RSS.* Also, obtained the Bayesian predictive distribution of the range when the parent distributions is one-parameter exponential.We have provided predicted intervals for future observations from one parameter exponential distribution in the case of *FSS*. Problems for constructing prediction limits have also been considered.

We have developed exact prediction intervals for failure times from one-parameter and two-parameter exponential distributions based on doubly Type-II censored samples and have developed exact prediction intervals for failure times of the items censored at the last observation from one-parameter and two-parameter exponential distributions based general progressively Type-II cenored samples.

In this report, we consider the Bayesian prediction for future observations from exponential distribution using both *FSS* and *RSS*. Motivation for this work is that in many biological and quality control problems, sample sizes cannot be taken as fixed all the times.

Let $x_{1:n} \le x_{2:n} \le ... \le x_{r:n}$, be the first r ordered lifetimes fail in a sample of $n(r \le n)$ components from the exponential distribution with *pdf*

$$f(x|\theta) = \theta e^{-\theta x}, \quad \theta, x > 0, \tag{1}$$

and let $x_{r+1:n} \le x_{r+2:n} \le ... \le x_{n:n}$, be the remaining $(n-r)$ lifetimes from the same distribution.

The joint density function of $X_{r:n}$ and $X_{s:n}$, $r < s$ is

$$f_{r,s:n}(x,y) = C_{r,s:n} F(x)^{r-1} (F(y) - F(x))^{s-r-1} (1 - F(y))^{n-s} f(x) f(y) \tag{2}$$

Where $f(.)$ and $F(.)$ are respectively the *pdf* and *cdf* of the exponential distribution given in (1) and

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \tag{3}$$

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Two different statistics to predict the future observation $x_{s:n}$ base on $x_{1:n} \le x_{2:n} \le ... \le x_{r:n}$, when the sample size n is fixed.

Their classical approach was based on the following two statistics

$$\text{Statistic-1 : } W = \frac{X_{s:n} - X_{r:n}}{S_r}, \tag{4}$$

$$\text{Statistic-2 : } U = \frac{X_{s:n} - X_{r:n}}{S_r}, \tag{5}$$

where $$S_r = \sum_{i=1}^{r} X_{i:n} + (n-r)X_{r:n}, \tag{6}$$

The distributions of Statistic-1 and Statistic-2 given in (4) and (5), require the pdf of $Z = X_{s:n} - X_{r:n}$

which is given by

$$\mathrm{f}(z|\theta) = \frac{\theta(e^{-\theta z})^{n-s+1}(1-e^{-\theta z})^{s-r-1}}{\beta(s-r, n-s+1)} \tag{7}$$

The goal is to predict the future order statistics based on the observed sample, $x_{1:n} \le x_{2:n} \le ... \le x_{r:n}$, using both FSS and RSS cases. In the case of FSS, the Bayes predictive density function of $y = x_{s:n}$ for given $x = \left(x_{1:n}, x_{2:n, \dots\dots\dots,} x_{r:n} \right)$ can be written as

$$h(y|x) = \int_0^\infty f(y|\theta)\, p(\theta|x) d\theta, \tag{8}$$

Where $f(y|\theta)$ is the conditional pdf of the future observation and $p(\theta|x)$ is the posterior pdf.

In the case of RSS, the predictive distribution function of y when the sample size n is a random variable is given by

$$q(y|x,n) = \frac{1}{\Pr(n \ge s)} \sum_{n=s}^{\infty} r(n) h(y|x), \tag{9}$$

Where $r(n)$ is the pmf of n and $h(y\,|\,x)$ is given in(8).

1. Prediction for Fixed Sample Size

In this section, we derive the predictive distribution function(8) based on the two different statistics given in (4) and (5). The posterior pdf $p(\theta|x)$ in (8) requires different priors. In the case of Statistic-1, the gamma pdf will be a suitable while the pdf of the $r-th$ order statistic will be suitable prior for Statistic-2. Also, obtaining the posterior requires the likelihood function which is

$$\mathrm{L}(x|\theta) = \frac{n!}{(n-r)!} \theta^{\mathrm{r}} \mathrm{e}^{-\theta \mathrm{S_r}} \tag{10}$$

Where S_r is given by (6).

1.1 Prediction Based on Statistic-1

Under Statistic-1, we assume the 2-parameter gamma *pdf* as a prior for the exponential parameter θ as

$$g(\theta|a,b)=\frac{1}{\Gamma(a)b^{a}}\theta^{a\text{-}1}e^{-\theta/b}\ ,\quad a,\text{b}\ >0 \tag{11}$$

Where a and b are known , Combining (11) with the likelihood function in (10), we obtain then the posterior pdf of θ as

$$p(\theta|x)=\frac{1}{\Gamma(R)}\theta^{R\text{-}1}A^{R}e^{-\theta A}\ , \tag{12}$$

where $\quad R=r+a, A=S_r+1/b,$ (13)

And S_r is given by (6).

The *pdf* of W given in (4) can be derived by using the transformation $W=Z/S_r$. This gives

$$f_w(w|\theta)=\ f_z(wS_r|\theta)\left|\frac{\partial Z}{\partial W}\right|$$

$$=\frac{S_r\theta(e^{-\theta wS_r})^{n-s+1}(1-e^{-\theta wS_r})^{s-r-1}}{\beta(s-r,n-s+1)} \tag{14}$$

From (8), the predictive function $h_1(w|x)$ can be written as

$$h_1(w|x)=\int_0^{\infty} f_w(w|\theta)\,p(\theta|x)d\theta, \tag{15}$$

Where $f_w(w|\theta)$ and $p(\theta|x)$ are given in (14) and (12) respectively.

By using (14) in (15) and expanding the term $(1-e^{-\theta wS_r})^{s-r-1}$ binomially and integrating out of θ, we get

$$h_1(w|x)=\ \frac{S_rRA^{R}}{\beta(s-r,n-s+1)}\sum_{i=0}^{s-r-1}\frac{\binom{s-r-1}{i}(-1)^{i}}{(A+(n-s+i+1)wS_r)^{R+1}}\ , \tag{16}$$

$$h_1(w|x)=\ \frac{RA^{R}}{\beta(s-r,n-s+1)}\sum_{i=0}^{s-r-1}\frac{\binom{s-r-1}{i}(-1)^{i}}{(A+(n-s+i+1)w)^{R+1}}\ , \tag{17}$$

Which upon using the transform $wS_r\ =\ y$ gives the same distribution given in (17). In fact both of (16) and (17) give the same cumulative distribution function given in (18).

$$H_1(t)=\Pr(W\le t|x)$$

$$=\ 1-\frac{A^{R}}{\beta(s-r,n-s+1)}\sum_{i=0}^{s-r-1}\frac{\binom{s-r-1}{i}(-1)^{i}(n-s+i+1)^{-1}}{(A+(n-s+i+1)t)^{R}}\ , \tag{18}$$

Where A and R are given in (13). The percentage points of the predictive *cdf* given in (18) can be easily obtained by solving the following nonlinear equation H1(t)= 1-α.. (19).

Then the exact two sided (1-α)100% Bayesian interval for the future observation $x_{s:n}$ can be constructed as

$$(t_{\alpha/2} + x_{r:n}, t_{1-\alpha/2} + x_{r:n}), \tag{20}$$

where $t_{\alpha/2}$ and $t_{1-\alpha/2}$ are the lower and upper percentage points of $H_1(t)$.

Example (1)

In this example, we generate 5 order statistics from the exponential *cdf* given in (1) as: 0.017,0.363,0.365,0.438 and 0.456. By using these data, we construct a 90% and a 95% predictive intervals for the observation $x_{s:n}$, n=10,s=6;7;8;9;10 by (20),these are given below:

r	s	90% P.I	95% P.I
5	6	(0.4591 , 0.7246)	(0.4574 , 0.8131)
5	7	(0.4803 , 0.9697)	(0.4723 , 1.1166)
5	8	(0.5180 , 1.2877)	(0.5019 , 1.5106)
5	9	(0.5765 , 1.7695)	(0.5494 , 2.1117)
5	10	(0.6815 , 2.7888)	(0.6347 , 3.4048)

Notice that , the values of a and b in the prior distribution are chosen randomly.

1.2 Prediction Based on Statistic-2

Under Statistic-2, we suggest the following prior *pdf* for the exponential parameter θ:

$$g(\theta) = \frac{n!}{(r-1)!(n-r)!}\theta(1-e^{-\theta x})^{r-1}e^{-(n-r+1)\theta x} \tag{21}$$

From (10) and (21), the posterior *pdf* of θ is obtained as

$$p(\theta|x) = \frac{1}{K\Gamma(r+2)}\theta^{r+1}(1-e^{-\theta x})^{r-1}e^{-\theta(S_r+(n-r+1)x)} \tag{22}$$

where $$K = \sum_{\ell=0}^{r-1}\frac{\binom{r-1}{\ell}(-1)^{\ell}}{(S_r+(n-r+\ell+1)x)^{r+2}} \tag{23}$$

The *pdf* of U given in (5) can be derived by using the transformation $U = Z / x_{r:n}$. This gives

$$f_U(u|\theta) = f_z(ux_{r:n}|\theta)\left|\frac{\partial Z}{\partial U}\right|$$

$$= \frac{x_{r:n}\theta(e^{-\theta ux_{r:n}})^{n-s+1}(1-e^{-\theta ux_{r:n}})^{s-r-1}}{\beta(s-r,n-s+1)} \tag{24}$$

From (8), the predictive function $h_2(u|x)$ can be written as

$$h_2(u|x) = \int_0^{\infty} f_U(u|\theta)p(\theta|x)d\theta, \tag{25}$$

Where $f_U(u|\theta)$ and $p(\theta|x)$ are given in (24) and (22) respectively.

By using (24) in (25) and expanding the term $(1-e^{-\theta x_{r:n}})^{r-1}$ and $(1-e^{-\theta u x_{r:n}})^{s-r-1}$ binomially and simplifying, we get

$$h_2(u|x) = \frac{x_{r:n}(r+2)}{K\beta(s-r,n-s+1)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \binom{r-1}{i}\binom{s-r-1}{j}$$

$$\times \frac{(-1)^{j+i}}{[S_r+(n-r+i+1)x+(n\_s+j+1)ux_{r:n})^{r+3}} \tag{26}$$

$$h_2(u|x) = \frac{(r+2)}{K\beta(s-r,n-s+1)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \binom{r-1}{i}\binom{s-r-1}{j} \times \frac{(-1)^{j+i}}{[S_r+(n-r+i+1)x+(n\_s+j+1)u)^{r+3}}, \tag{27}$$

Which upon using the transform $wS_r = y$ gives the same distribution given in (27).
In fact both of (25) and (27) give the same cumulative distribution function given in (28).

$$H_2(t) = \Pr(U \le t|x)$$

$$= \frac{1}{K\beta(s-r,n-s+1)} \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1} \frac{\binom{s-r-1}{j}\binom{r-1}{i}(-1)^{j+i}}{(n-s+j+1)}$$

$$\times\left[\frac{1}{[S_r+(n-r+i+1)x+(n-s+j+1)t]^{r+2}} - \frac{1}{[S_r+(n-r+i+1)x]^{r+2}}\right] \tag{28}$$

Where S_r and K are given by(6) and (23) respectively.

The percentage points of the predictive cdf given in (18) can be easily obtained by solving the following nonlinear equation $H_2(t)$ = 1- α (29)

Then the exact two sided (1-∝) 100% Bayesian interval for the future observation $x_{s:n}$ is given in (20), where $t_{\propto/2}$ and $t_{1-\propto/2}$ in this case are the lower and upper percentage points of $H_2(t)$.

Example (2)

In this example, we use the same sample as in Example (1) for construct a 90% and a 95% predictive intervals for the observation $x_{s:n}$, n=10,s=6;7;8;9;10 by (20),(28)and (29).The result are given below:

R	s	90% P.I	95% P.I
5	6	(0.4608 , 0.7781)	(0.4584 , 0.8659)
5	7	(0.4920 , 1.0507)	(0.4804 , 1.1839)
5	8	(0.5490 , 1.3997)	(0.5256 , 1.5914)
5	9	(0.6388 , 1.9312)	(0.5996 , 2.2189)
5	10	(0.8002 , 3.0781)	(0.7328 ,3.6004)

2. Prediction for Random Sample Size

In this section, we assume the sample size to be random and distributed as(i) Poisson distribution and (ii) binomial distribution.

2.1 Sample Size has Poisson Distribution

We assume here that, the sample size n has Poisson distribution with *pmf* given by

$$p(n;\theta)=\frac{e^{-\theta}\theta^{n}}{n!}\ ,\ n=0,1,2,...,\theta>0 \tag{30}$$

Replacing $r(n)$given in(9) by $p(n;\theta)$given in(30) , we obtain

$$q(y|x,n)=\frac{1}{1-P(s-1)}\sum_{n=s}^{\infty}\frac{e^{-\theta}\theta^{n}}{n!}h(y|x), \tag{31}$$

Where P(.) is the *cdf* of the Poisson distribution.

In the following two subsections, we use(31) to derive the Bayesian redictive *pdf* in the case of *RSS* based on both Statistic-1 and Statistic-2.

2.1.1 Prediction Based on Statistic-1

Using Statistic-1 and replacing $h(y\,|\,x)$ given in (31) by $h_1(w\,|\,x)$ given in (17) , we obtain the Bayes predictive *pdf* of W when the sample size is distributed as $p(n;\theta)$.

Starting from the predictive function in the case of *RSS* given in (9), we can write the predictive function of Stataistic-1 based on *RSS* as

$$q_1(w|x,n)=\frac{1}{1-P(s-1)}\sum_{n=s}^{\infty}p(n)h_1(w|x), \tag{32}$$

where $p(.)$ and $P(.)$ are the pmf snd cdf of the Poisson distribution respectively and $h_1(.|x)$ is given in (17) . Upon using(17) , (30) and intergrationg out of θ , we get (33),

$$q_1(w|x,n)=\frac{RA^{R}e^{-\theta}}{1-P(s-1)}\sum_{n=s}^{\infty}\sum_{i=0}^{s-r-1}\frac{\theta^{n}\binom{s-r-1}{i}(-1)^{i}[A+(n-s+i+1)w]^{-R-1}}{n!\beta(s-r,n-s+1)}, \tag{33}$$

The *cdf* of W is given by

$$Q_1(t)=\Pr(W\le t\,|\,x)$$

$$=1-\frac{1}{1-P(s-1)}\sum_{n=s}^{\infty}\sum_{i=0}^{s-r-1}\frac{A^{R}e^{-\theta}\theta^{n}\binom{s-r-1}{i}(-1)^{i}}{n!\beta(s-r,n-s+1)(n-s+i+1)}$$

$$\times\left[\frac{1}{[A+(n-s+i+1)t]^{R}}\right], \tag{34}$$

where A and R are given in (13).

The percentage points of the predictive *cdf* given in (34) can be easily obtained by solving the following nonlinear equation $Q_1\,(t)=1\text{-}\alpha$ (35)

Then the exact two sided(1-∝) 100% Bayesian interval for the future observation $x_{s:n}$ is given in (20), where $t_{\propto/2}$ and $t_{\,1\text{-}\propto/2}$ in this case are the lower and upper percentage points of $Q_1\,(t)$.

Example (3)

In this example , we generate the sample size n from $p(n{:}2)$ to be7. Then, we generate the first 6 order statistics based on n=7 from the standard exponential *pdf* they are:

0.153,0.417,0.433,0.720,0.876 and 0.926. Next, this sample is used to predict the 90% and 95% predictive intervals for the future observations up to 10 as give below:

r	s	90% P.I	95% P.I
6	7	(0.9520 , 3.1091)	(0.9390 , 3.7941)
6	8	(1.0538 , 3.7831)	(1.0118 , 4.5553)
6	9	(1.1603 , 4.2142)	(1.0985 , 5.0401)
6	10	(1.2547 , 4.5361)	(1.1796 , 5.4022)

2.1.2 Prediction Based on Stataistic-2

Using Statistic-2 and replacing $h(y\,|\,x)$ given in (31) by $h_2(u\,|\,x)$ given in (27) , we obtain the Bayes predictive pdf of U when the sample size is distributed as $p(n;\theta)$.

Follow by the above calculation , we can obtained (36)

$$q_2(u|x,n)=\frac{r+2}{1-P(s-1)}\sum_{n=s}^{\infty}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\frac{e^{-\theta}\theta^{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{Kn!\,\beta(s-r,n-s+1)}\times\frac{1}{[S_r+(n-s+i+1)x+(n-s+j+1)u]^{r+3}} \qquad (36)$$

The cdf of U is given by

$$\begin{aligned}Q_2(t)&=\Pr(U\le t\,|\,x)\\&=\frac{1}{1-P(s-1)}\sum_{n=s}^{\infty}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\frac{e^{-\theta}\theta^{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{Kn!(n-s+j+1)}\\&\quad\times\left[\frac{1}{[S_r+(n-s+i+1)x+(n-s+j+1)t]^{r+2}}\right.\\&\quad\left.-\frac{1}{[S_r+(n-s+i+1)x]^{r+2}}\right],\end{aligned} \qquad (37)$$

where S_r and K are given in (6)and (23), respectively.

The percentage points of the predictive cdf given in (37) can be easily obtained by solving the following nonlinear equation $Q_2\,(t) = 1-\alpha$ (38)

Then the exact two sided(1-∝) 100% Bayesian interval for the future observation $x_{s:n}$ is given in (20), where $t_{\propto/2}$ and $t_{1-\propto/2}$ in this case are the lower and upper percentage points of $Q_2\,(t)$.

Example (4)

By using the same data in Example (3) ,we predict the 90% and 95% predictive intervals for the future observations up to 10based on Statistic-2 when the sample size is *p(n;2)*. The results are give below:

R	s	90% P.I	95% P.I
6	7	(0.9494 , 2.6086)	(0.9379 , 3.0815)
6	8	(1.0591 , 3.3841)	(1.0157 , 3.9823)
6	9	(1.1972 , 4.0648)	(1.1264, 4.7787)
6	10	(1.3485 , 4.6963)	(1.2596 , 5.4864)

2.2 Sample Size has the Binomial Distribution

In this section , we assume that the sample size *n* has the binomial distribution, *b(n;M;p)*, with *pmf*

$$b(n;M,p)=\binom{M}{n}p^nq^{M-n}, \qquad q=1-p, n=0,1,2,...,M. \tag{39}$$

Replacing $r(n)$ given in (9) by $b(n;M,p)$ given in (39) we obtain

$$v(y\,|\,x,n)=\frac{1}{1-B(s-1)}\sum_{n=s}^{M}\binom{M}{n}p^nq^{M-n}h(y\,|\,x), \tag{40}$$

where *B(.)* is the *cdf* of the binomial distribution. In the following two subsections, we use (40) to derive the Bayesian predictive *pdf* in the case of *RSS* based on both Statistic-1 and Statistic-2.

2.2.1 Prediction Basedon Statistic-1

Using Statistic-1 and replacing $h(y\,|\,x)$ given in (40) by $h_1(w\,|\,x)$ given in (17) , we obtain the Bayes predictive *pdf* of W when the sample size is distributed as $b(n;M,p)$. This is given by

$$v_1(w\,|\,x,n)=\frac{RA^R}{1-B(s-1)}\sum_{n=s}^{M}\sum_{j=0}^{s-r-1}\frac{(-1)^j\binom{M}{n}\binom{s-r-1}{j}[A+(n-s+j+1)w]^{-R-1}}{p^{-n}q^{n-M}\beta(s-r,n-s+1)} \tag{41}$$

The *cdf* of W is given by

$$V_1(t)=\Pr(W\le t\,|\,x)$$

$$=1-\frac{1}{1-B(s-1)}\sum_{n=s}^{M}\sum_{j=0}^{s-r-1}\frac{A^R\binom{M}{n}p^nq^{M-n}\binom{s-r-1}{j}(-1)^j}{(n-s+j+1)\beta(s-r,n-s+1)}$$

$$\times\frac{1}{[A+(n-s+j+1)t]^R} \tag{42}$$

where A and R are given in (13)

Example (5)

In this example, we generate the sample size *n* from *b(n;15,0.3)*. then based on the generated *n*, we genetate the first 6 order statistics from the standard exponential as: 0.177,0.159,0.493,1.262,1.376 and 1.444. By using this data we predict the 90% and 95% predictive intervals for the future observations up to 10 ,based on Statistic-1 when the sample size is *b(n;15,0.3)*. This is given below:

R	s	90% P.I	95% P.I
6	7	(1.4724 , 4.0975)	(1.4581 , 5.0142)
6	8	(1.5975 , 5.0538)	(1.5473 , 6.0555)
6	9	(1.7426 , 5.6908)	(1.6646 , 6.7684)
6	10	(1.8745 , 6.1762)	(1.7738 , 7.3189)

2.2.2 Prediction Basedon Statistic-2

Again using Statistic-2 and replacing $h(y\,|\,x)$ given in (40) by $h_2(u\,|\,x)$ given in (27) , we obtain the Bayes predictive *pdf* of U when the sample size is distributed as $b(n;M,p)$. This is given by

$$v_2(u \mid x,n) = \frac{r+2}{1-B(s-1)} \sum_{n=s}^{M} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{M}{n}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{K\beta(s-r,n-s+1)} \times \frac{1}{[S_r+(n-s+i+1)x+(n-s+j+1)u]^{r+3}} \tag{43}$$

The *cdf* of U is given by

$$V_2(t) = \Pr(U \le t \mid x)$$

$$= \frac{1}{1-B(s-1)} \sum_{n=s}^{M} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} \frac{\binom{M}{n} p^n q^{M-n} \binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}}{K\beta(s-r,n-s+1)(n-s+j+1)} \times \left[\frac{1}{[S_r+(n-r+i+1)x+(n-s+j+1)t]^{r+2}} - \frac{1}{(S_r+(n-r+i+1)x)^{r+2}} \right],$$

where S_r and K are given in (6) and (23), respectively.

Example (6)

By using the same data in Example (5),we predict the 90% and 95% predictive intervals for the future observations up to 10 ,based on Statistic-2 when the sample size is *b(n;15,0.3)*. This is given below:

R	s	90% P.I	95% P.I
6	7	(1.4724 , 3.7642)	(1.4589 , 4.4340)
6	8	(1.6288 , 4.9745)	(1.5701 , 5.8308)
6	9	(1.8394 , 6.0728)	(1.7394 , 7.1167)
6	10	(2.0763 , 7.0983)	(1.9372 , 8.2842)

3. Application

In this sectiioin, we apply our technique to some real data which follow the exponential distribution as presented in Lawless in which the test is terminated after the following 4 failures: 30,90,120 and 170 hours.By using these four times, we calculate the percentage point for the predictive furctions presented in Section 2 and 3 based on *FSS* and *RSS* u to 10 failures as given below:

Table 1. Percentage points based on Statistic-1 when n=10 is fixed.

r	S	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	0.5	1.2	2.5	5.1	139.4	195.8	260	360.3
4	6	7.2	11.8	17.5	26.7	273.6	365.3	469.9	630.4
4	7	22.1	31.9	43	59.5	435.4	570.2	722.7	957.9
4	8	45.5	61.7	79.4	105	650.3	843.2	1061.1	1396.1
4	9	80.8	106	133	171.6	976.6	1261.6	1582.1	2077.8
4	10	141.6	183	227.2	290.6	1655	2146.4	2703.3	3565.9

Table 2. Percentage points based on Statistic -2 when n=10 is fixed.

r	S	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	0.5	1.3	2.7	5.6	138.6	187.8	241	317.9
4	6	8.2	13.5	19.9	30	266	341	420.8	534.4
4	7	25.9	37.1	49.6	67.9	417.8	523.4	634.9	793.4
4	8	54	72.7	92.6	120.8	618.8	766	921	1141
4	9	97	125.9	156.1	198.5	925.1	1139.6	1365.6	1687
4	10	171.4	218.3	267.5	336.4	1569.7	1946.4	2346.8	2919.6

Table 3. Percentage points based on Statistic-1 when *n~p(n;2)* .

r	S	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	2	5.1	10.4	21.7	707.7	1014.8	1370	1926.7
4	6	21	34.9	52.4	81.3	1002.3	1367.5	1786	2436
4	7	48.4	70.9	97	137.1	1196.8	1599.3	2058.6	2770.3
4	8	75.8	104.6	136.8	184.8	1343.7	1773.5	2263.3	3022.2
4	9	101.2	134.7	171.5	225.5	1461.7	1915.4	2430.6	3224.2
4	10	124.1	161.6	201.9	260.5	1560.6	2033.2	2569.4	3397.1

Table 4. Percentage points based on Statistic-2 when *n~p(n;2)* .

r	S	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.6	4.2	8.5	17.6	501.5	690.6	899.2	1223.8
4	6	19.2	31.8	47.2	72.6	753.2	982.9	1233.7	1589.4
4	7	48	69.7	95.6	132.3	953.9	1226.1	1498.9	1974
4	8	81	113.1	144.8	192	1141.6	1446.4	1767.7	2255.2
4	9	118.3	153.4	194.3	251.3	1318.3	1646.4	2016.7	2555.4
4	10	152.4	198.2	244.2	308	1490.6	1846.5	2241.9	2815.6

Table 5. Percentage points based on Statistic-1 when *n-b(n;15,0.3).*

r	S	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.4	3.7	7.5	15.6	569.3	844.9	1148.2	1602.3
4	6	16.9	28.3	42.3	66.3	887.3	1229.1	1613	2235.4
4	7	42.3	62.4	86.5	121.5	1115.2	1506.7	1930.8	2548.4
4	8	70.3	98.2	128	172.7	1288.2	1712.5	2163.9	2901.1
4	9	98.5	132.7	167.1	219.8	1434.4	1886.4	2395.7	3281.3
4	10	124.6	160.5	202.3	260.8	1556.2	2044.7	2590.3	3487.2

Table 6. Percentage points based on Statistic-2 when *n-b(n;15,0.3).*

r	S	1%	2.5%	5%	10%	90%	95%	97.5%	99%
4	5	1.2	3.2	6.5	13.5	411.2	528.4	776.2	1056.9
4	6	16.3	27	40	61.7	674.6	895.4	1139.4	1492.5
4	7	43.6	63.3	86.8	120.3	895.9	1159.1	1439.1	1831.7
4	8	76.9	107.3	137.4	182.4	1103.4	1402.5	1720.6	2193.2
4	9	115.8	150.2	190.2	246.1	1299	1629.9	1965.5	2506.7
4	10	155.1	198.5	245.4	308.1	1487.4	1854.9	2254.8	2822.2

From the above Tables 1-6, we find the 90% and 95% predictive intervals for the fifth failure *FSS* and *RSS* when $n \sim p(n;2)$ and $n\text{-}b(n;15,0.3)$.using (17) . These are given below:

1-∝	Sample size	Statistic-1	Statistic-2
90%	Fixed*(n=10)*	(172.5 , 365.8)	(172.7 , 357.8)
	p(n;2)	(180.4 , 1184.8)	(178.5 , 860.6)
	(n;15,0.3)	(177.5 , 1115)	(176.5 , 752.4)
95%	Fixed*(n=10)*	(171.2 , 430.0)	(171.3 , 411.0)
	p(n;2)	(175.1 , 1540.0)	(174.2 , 1069.2)
	(n;15,0.3)	(173.7 , 1318.2)	(173.2 , 946.2)

From the above table we see that the width of the Bayesian predictive intervals based on random and fixed simple sizes are close for most cases.

To show the efficiency of our Bayesian technique, we calculate the width of the 95% predictive intervals the results given in the above table analogues with those of the classical technique by Lawless and Lingappaiah as given below:

	Classical Approach	Bayesian Approach
Statistic-1	262.6	258.8
Statistic-2	261.9	239.7

It is clear that the Bayesian approach gives narrower predictive intervals than the corresponding classical intervals.

Conclusion

The Baysian predictive function for future observations from the exponential distribution based on the fixed and random sample size are investigated. For the finite populations, the binomial distribution is considered to be a suitable model for the sample sizes, while for the large populations, Poisson distribution can be considered as a suitable model To show the usefulness of the proposed procedure simulation expriments are carried out and an application is discussed. Finally, we conclude the following remarks:

1. The random samples from the exponential distribution are generated by using the double precision subroutine RNEXP from the IMSL library.
2. The nonlinear equations are solved by using the double precision subroutine ZREAL from the IMSL library.
3. The parameters of the prior a and b are positive values and selected randomly which do not affect the calculations.
4. The sample size n is generated randomly from binomial and Poisson distributions by using the subroutines RNBIN and RNOPI, respectively, from the IMSL library.

5. The proportion of future responses that can be predicted using the proposed predictive intervals is investigated in the sense of probablity coverage. It gives probabilities close to their significance levels
6. The Bayesian approach gives better results than the classical approach in the sense of the predictive average width of the predictive intervals.

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