Weak Solutions of the Linear Thermoelectric Magnetohydrodynamics Convection Problem

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Abstract

We consider the linear case of Thermoelectric Magnetohydrodynamic Convection problem in three dimensional space. We prove the existence and uniqueness of the solutions with weak sense in some Hilbert spaces.

Key words: Thermoelectric Magnetohydrodynamic Convection, Weak sense, Hilbert space

Introduction

The main purpose of this research is to study the weak solutions of TEMHD problem. In the first section, we describe some function spaces, definitions and notations for this problem. The governing equations of the problem are presented in the second section. The variational formulation of the problem is described in the third section. Finally, we described the existence and uniqueness of the weak solutions in the fourth section.

Preliminaries and Notations

Let Ω be the open bounded subset of $\mathbb{R}^3$. Let a, b be two extended real numbers and $-\infty \le a \le b \le \infty$. If X be a Banach space, for given α, $1 \le \alpha < \infty$,

$$L^\alpha(a, b; X) = \{f : [a,b] \to X : \left(\int_a^b \|f(t)\|_X^\alpha dt\right)^{\frac{1}{\alpha}} < \infty\},$$

$$L^\infty(a, b; X) = \{f : [a,b] \to X : \text{Ess sup} \|f(t)\|_X < \infty\},$$

$$C(\Omega) = \{f : f : \Omega \to K \text{ is linear and continuous}\},$$

$$C^m(\Omega) = \{f \in C(\Omega) : D^\alpha f \in C(\Omega), \forall \alpha, |\alpha| \le m, m > 0\}.$$

Let $\mathcal{D}(\Omega)$ denotes the space consisting of infinitely differentiable functions with compact support in Ω. We also denote

$\mathcal{V} = \{u \in \mathcal{D}(\Omega), \text{div } u = 0\}$,

V = the closure of $\mathcal{V}$ in $H_0^1(\Omega)$,

H = the closure of $\mathcal{V}$ in $L^2(\Omega)$,

$\mathcal{W} = \mathcal{D}(\Omega)$,

W = the closure of $\mathcal{W}$ in $H_0^1(\Omega)$ and

G = the closure of $\mathcal{W}$ in $L^2(\Omega)$.

Let V', W', H' and G' denote the dual spaces of V, W, H and G.

Then we have the inclusions $V \subseteq H \equiv H' \subseteq V'$ and $W \subseteq G \equiv G' \subseteq W'$.

As a consequence of the previous identifications, $\langle f, u \rangle = (f, u)$, for all $f \in H$, for all $u \in V$.

Let p be a distribution on Ω, $p \in \mathcal{D}'(\Omega)$. It is easy to check that for any $v \in \mathcal{V}$, we have

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$$< \text{grad } p, v > = \sum_{i=1}^{n} < D_i p, v_i > = -\sum_{i=1}^{n} < p, D_i v_i > = < p, \text{div } v > = 0.$$

For fixed u in V, the mapping $V \to \mathbb{R}$, $v \to ((u, v))$ is linear and continuous on V. Then, there exists an element of V', denote Au such that $< Au, v > = ((u, v))$, for all $v \in V$. Then $u \to Au$ is linear and continuous and also isomorphism from V to V'.

Variational Formulation of Linear TEMHD Problem

Consider the linear TEMHD Problem (Straughan, 1992)

$$\begin{aligned}
&\frac{du}{dt} - \nu \Delta u + \frac{1}{\rho}\nabla p = \lambda_1 \overline{H}(h_{,z} - \nabla h_3) + \lambda_2 \theta \delta_{i3}, \\
&\frac{dh}{dt} - \eta \Delta h = \overline{H} \nabla u . \delta_{i3}, \\
&\frac{d\theta}{dt} - \kappa \Delta \theta = \beta w + \lambda_3 \,\text{curl}\, h . \delta_{i3}, \\
&\nabla . u = 0, \text{in } Q = \Omega \times [0, T], \\
&\nabla . h = 0, \text{in } Q = \Omega \times [0, T],
\end{aligned} \tag{1}$$

where u = (u, v, w): the velocity, p: the pressure, θ : the temperature, h = (h_1, h_2, h_3): the intensity of the magnetic field.

We define the periodicity cell, domain and function space in the form
$\Omega = \{(x, y, z) \in (0, P_1) \times (0, P_2) \times (0, d)\}$, $Q = \Omega \times [0, T]$ and $\partial\Omega$, the boundary of Ω.
We may be written the system (1) as

$$\begin{aligned}
&\frac{du}{dt} - \nu \Delta u + \frac{1}{\rho}\nabla p = f_1, \\
&\nabla . u = 0, \\
&\frac{dh}{dt} - \eta \Delta h = f_2, \\
&\nabla . h = 0, \\
&\frac{d\theta}{dt} - \kappa \Delta \theta = f_3,
\end{aligned} \tag{2}$$

with the boundary conditions

$$u = 0,\ h = 0,\ \theta = 0,\ p = 0 \ \text{ on } = \partial\Omega \times [0, T], \tag{3}$$

the periodic boundary conditions u, p, h, θ are periodic in x and y direction with period P_1 in x direction and P_2 in y direction respectively and the initial conditions

$$u(x, 0) = u_0(x),\ h(x, 0) = h_0(x, 0) \text{ and } \theta(x, 0) = \theta_0(x) \ \text{ in } \Omega. \tag{4}$$

Here $f_1 = \lambda_1 \overline{H}(h_{,z} - \nabla h_3) + \lambda_2 \theta \delta_{i3}$, $f_2 = \overline{H} \nabla u . \delta_{i3}$ and $f_3 = \beta w + \lambda_3 \,\text{curl}\, h . \delta_{i3}$.

Suppose that u, p, θ and h are classical solutions of the system (2)-(4) and $u, h \in (C^2(\overline{Q}))^3$, $\theta \in C^2(\overline{Q})$ and $p \in C^1(\overline{Q})$.

Now, we will consider the variational formulation of the given problem.

Let $v_1, v_2 \in \mathcal{V}$ and $r \in \mathcal{W}$. Multiplying $(2)_1$ by v_1, $(2)_2$ by v_2 and $(2)_3$ by r and integration over Ω, we obtain

$$(u_t, v_1) - \nu(\Delta u, v_1) + \frac{1}{\rho}(\nabla p, v_1) = (f_1, v_1),$$
$$(h_t, v_2) - \eta(\Delta h, v_2) = (f_2, v_2), \quad (5)$$
$$(\theta_t, r) - \kappa(\Delta\theta, r) = (f_3, r).$$

Using $(2)_2$ and $(2)_4$ and by continuity, the system (5) can be written as

$$\frac{d}{dt}(u, v_1) + \nu((u, v_1)) = (f_1, v_1),$$
$$\frac{d}{dt}(h, v_2) + \eta((h, v_2)) = (f_2, v_2),$$
$$\frac{d}{dt}(\theta, r) + \kappa((\theta, r)) = (f_3, r).$$

Now, we obtain following weak formulation of the problem.

Problem (1)

Let $v_1, v_2 \in \mathcal{V}$ and $r \in \mathcal{W}$.

Let $u_0, h_0 \in H$, $\theta_0 \in G$.

To find u, h and θ satisfying $u, h \in L^2(0, T; V)$, $\theta \in L^2(0,T;W)$ and satisfying the equations

$$\frac{d}{dt}(u, v_1) + \nu((u, v_1)) = \langle f_1, v_1 \rangle, \quad (6)$$
$$\frac{d}{dt}(h, v_2) + \eta((h, v_2)) = \langle f_2, v_2 \rangle, \quad (7)$$
$$\frac{d}{dt}(\theta, r) + \kappa((\theta, r)) = \langle f_3, r \rangle, \quad (8)$$

with the initial conditions $u(0) = u_0$, $h(0) = h_0$ and $\theta(0) = \theta_0$. (9)

The spaces $L^2(0,T;V)$, $L^2(0,T;W)$, H, G, $L^2(0,T;V')$ and $L^2(0,T;W')$ are the spaces for which existence and uniqueness of the weak solutions will be proved.

For linear case, suppose that $u, h \in L^2(0,T;V)$ and $\theta \in L^2(0,T;W)$.

Then $A_1u, A_2h \in L^2(0,T;V')$, $A_2\theta \in L^2(0,T;W')$. Hence $f_1 - \nu A_1u$, $f_2 - \eta A_2h \in L^2(0,T;V')$ and $f_3 - \kappa A_2\theta \in L^2(0,T;W')$.

Then, we get

$$\frac{du}{dt} = (f_1 - \nu A_1 u)(t),$$
$$\frac{dh}{dt} = (f_2 - \eta A_1 h)(t),$$
$$\frac{d\theta}{dt} = (f_3 - \kappa A_2 \theta)(t).$$

So, we can see that $u', h' \in L^2(0,T;V')$ and $\theta' \in L^2(0,T;W')$. (10)

Also, $u:[0, T] \to V'$, $h:[0, T] \to V'$ and $\theta:[0, T] \to W'$ are absolutely continuous a. e.
In addition, the alternative formulation of the linear weak problem is the following:

Problem (2)

Let $u_0, h_0 \in H$ and $\theta_0 \in G$.

To find u, h and θ satisfying $u, h \in L^2(0, T;V)$, $\theta \in L^2(0,T;W)$ and satisfying the equations

$$u' + \nu A_1 u = f_1, \quad (11)$$

$$h' + \eta A_1 h = f_2, \quad (12)$$

$$\theta' + \kappa A_2 \theta = f_3, \quad (13)$$

with the initial conditions $u(0) = u_0$, $h(0) = h_0$, $\theta(0) = \theta_0$. (14)

We shall show that any solutions of problem (1) are the solutions of problem (2). The converse is also clear. Problems (1) and (2) are equivalent.

Construction of Approximate Solutions

We will use the Faedo-Galarkin method to construct the approximate problem (Teman, 1979).

Since V and W are separable, there exists the sequence of linearly independent elements x_i and y_i which are total in V and z_i which is total in W, i = 1, 2, 3, … m.

For each m, we define an approximate solutions of problem (1) and (2) as follows:

$$u_m = \sum_{i=1}^{m} u_{im}(t) x_i, \quad (15)$$

$$h_m = \sum_{i=1}^{m} h_{im}(t) y_i, \quad (16)$$

$$\theta_m = \sum_{i=1}^{m} \theta_{im}(t) z_i, \quad (17)$$

the functions u_{im}, h_{im}, θ_{im}, $1 \le i \le m$, are the scalar functions defined on [0, T].

Then $u'_m = \sum_{i=1}^{m} u'_{im}(t) x_i$, $h'_m = \sum_{i=1}^{m} h'_{im}(t) y_i$ and $\theta'_m = \sum_{i=1}^{m} \theta'_{im}(t) z_i$.

Assume that $f^1_{m-1} = \lambda_1 \overline{H}(h_{m-1,z} - \nabla h_{3(m-1)}) + \lambda_2 \theta_{m-1} \delta_{i3}$,

$f^2_{m-1} = \overline{H} \nabla u_{m-1} . \delta_{i3}$ and

$f^3_{m-1} = \beta w_{m-1} + \lambda_3 \, \mathrm{curl}\, h_{m-1} . \delta_{i3}$.

From the linear problem (1), we get

$$(u'_m, x_j) + \nu((u_m, x_j)) = \langle f^1_{m-1}, x_j \rangle, \quad (18)$$

$$(h'_m, y_j) + \eta((h_m, y_j)) = \langle f^2_{m-1}, y_j \rangle, \quad (19)$$

$$(\theta'_m, z_j) + \kappa((\theta_m, z_j)) = \langle f^3_{m-1}, z_j \rangle, \quad (20)$$

with the initial conditions $u_m(0) = u_{0m}$, $h_m(0) = h_{0m}$ and $\theta_m(0) = \theta_{0m}$, where u_{0m}, h_{0m} and θ_{0m} are orthogonal projections in H of u_0 on the space spanned by $x_1, x_2, x_3, \ldots, x_m$, H of h_0 on the space spanned by $y_1, y_2, \ldots, y_m$ and G of θ_0 on the space spanned by $z_1, z_2, z_3, \ldots, z_m$ respectively.

From, (18)-(20), we have

$$\sum_{i=1}^{m}(x_i,x_j)u'_{im}(t)+\nu\sum_{i=1}^{m}(x_i,x_j)u_{im}(t)=<f^1_{m-1}(t),x_j>,$$
$$\sum_{i=1}^{m}(y_i,y_j)h'_{im}(t)+\eta\sum_{i=1}^{m}(y_i,y_j)h_{im}(t)=<f^2_{m-1}(t),y_j>, \tag{21}$$
$$\sum_{i=1}^{m}(z_i,z_j)\theta'_{im}(t)+\kappa\sum_{i=1}^{m}(z_i,z_j)\theta_{im}(t)=<f^3_{m-1}(t),z_j>,1\le j\le m.$$

Since x_i, y_i and z_i, $1\le i\le m$ are linearly independent, the matrices $[(x_i, x_j)]_{1\le i\le m}$, $[(y_i, y_j)]_{1\le i\le m}$ and $[(z_i, z_j)]_{1\le i\le m}$ are nonsingular.

Using the inverse matrices to (21), we obtain

$$u'_{im}(t)+\sum_{j=1}^{m}\alpha_{ij}u_{jm}(t)=\sum_{j=1}^{m}\beta_{ij}<f^1_{m-1}(t),x_j>,$$
$$h'_{im}(t)+\sum_{j=1}^{m}\tilde{\alpha}_{ij}h_{jm}(t)=\sum_{j=1}^{m}\tilde{\beta}_{ij}<f^2_{m-1}(t),y_j>, \tag{22}$$
$$\theta'_{im}(t)+\sum_{j=1}^{m}\overline{\alpha}\theta_{jm}(t)=\sum_{j=1}^{m}\tilde{\beta}_{ij}<f^3_{m-1}(t),z_j>,1\le j\le m.$$

with the initial conditions

$u_{im}(0)$ = the i^{th} component of u_{0m},
$h_{im}(0)$ = the i^{th} component of h_{0m} (23)

and

$\theta_{im}(0)$ = the i^{th} component of θ_{0m}.

The linear differential system (22) together with the initial conditions (23) define uniquely on the whole interval [0, T].

Consider

$$\int_0^T\left|<f^1_{m-1},x_j>\right|^2dt=\int_0^T\left|<\lambda_1\overline{H}(h_{m-1,z}-\nabla h_{3(m-1)})+\lambda_2\theta_{m-1}\delta_{i3},x_j>\right|^2dt$$
$$\le\int_0^T\left\|\lambda_1\overline{H}(h_{m-1,z}-\nabla h_{3(m-1)})+\lambda_2\theta_{m-1}\delta_{i3}\right\|^2_{V'}dt.$$

Then $<f^1_{m-1},x_j>$ is squared integrable. u_{im}'s are the sum of squared integrable functions so u_{im}'s are also square integrable. Therefore, for each m, $u_m\in L^2(0,T;V)$ and $u_m'\in L^2(0,T;V')$.

Also, $\int_0^T\left|<f^2_{m-1},y_j>\right|^2dt=\int_0^T\left|<\overline{H}\nabla u_{m-1}.\delta_{i3},y_j>\right|^2dt\le\int_0^T\left\|\overline{H}\nabla u_{m-1}\right\|^2_{V'}dt$ and

$$\int_0^T\left|<f^3_{m-1},z_j>\right|^2dt=\int_0^T\left|<\beta w_{m-1}+\lambda_3\,\mathrm{curl}\,h_{m-1}.\delta_{i3},z_j>\right|^2dt\le\int_0^T\left\|\beta w_{m-1}+\lambda_3\,\mathrm{curl}\,h_{m-1}.\delta_{i3}\right\|^2_{W'}dt.$$

Obviously, these inequalities are bounded. Hence, the square functions from t to $<f^2_{m-1},y_j>$

and $< f^3_{m-1}, z_j >$ are square integrable and then h_{im} and θ_{im} are the sum of square integrable functions. So, for each m, $h_m \in L^2(0,T;V)$, $h_m' \in L^2(0,T;V')$, $\theta_m \in L^2(0,T;W)$ and $\theta_m' \in L^2(0,T;W')$.

Existence and Uniqueness of the Solutions

We will consider a priori estimates independent of m for the functions u_m, h_m and θ_m and then pass to the limit.

Lemma *If u_m, h_m and θ_m defined by* (15)-(17) *are approximate solutions of linear problem* (1), *then*

(i) *u_m remains in a bounded set of $L^\infty(0, T; H) \cap L^2(0, T; V)$,*

(ii) *h_m remains in a bounded set of $L^\infty(0, T; H) \cap L^2(0, T; V)$,*

(iii) *θ_m remains in a bounded set of $L^\infty(0, T; G) \cap L^2(0, T; W)$.*

Proof: Multiplying (18) by $u_{jm}(t)$, (19) by $h_{jm}(t)$ and (20) by θ_{jm} and add all these equations for $j = 1, 2, 3, \ldots, m$, we obtain

$$(u'_m(t), u_m(t)) + \nu((u_m(t), u_m(t))) = < f^1_{m-1}(t), u_m(t) >,$$

$$(h'_m(t), h_m(t)) + \eta((h_m(t), h_m(t))) = < f^2_{m-1}(t), h_m(t) >,$$

$$(\theta'_m(t), \theta_m(t)) + \kappa(\theta'_m(t), \theta_m(t)) = < f^3_{m-1}(t), \theta_m(t) >.$$

And hence,

$$\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 \le \frac{1}{\nu}\left\|f^1_{m-1}(t)\right\|^2_{V'}, \qquad (24)$$

$$\frac{d}{dt}|h_m(t)|^2 + \eta\|h_m(t)\|^2 \le \frac{1}{\eta}\left\|f^2_{m-1}(t)\right\|^2_{V'}, \qquad (25)$$

$$\frac{d}{dt}|\theta_m(t)|^2 + \kappa\|\theta_m(t)\|^2 \le \frac{1}{\kappa}\left\|f^3_{m-1}(t)\right\|^2_{W'}. \qquad (26)$$

Integrating (24)-(26) from 0 to T yields

$$|u_m(T)|^2 + \nu\int_0^T \|u_m(t)\|^2 dt \le |u_{0m}|^2 + \frac{1}{\nu}\int_0^T \left\|f^1_{m-1}(t)\right\|^2_{V'} dt,$$

$$|h_m(T)|^2 + \eta\int_0^T \|h_m(t)\|^2 dt \le |h_{0m}|^2 + \frac{1}{\eta}\int_0^T \left\|f^2_{m-1}(t)\right\|^2_{V'} dt,$$

$$|\theta_m(T)|^2 + \kappa\int_0^T \|\theta_m(t)\|^2 dt \le |\theta_{0m}|^2 + \frac{1}{\kappa}\int_0^T \left\|f^3_{m-1}(t)\right\|^2_{W'} dt.$$

Since $u_{0m} \to u_0$, $h_{0m} \to h_0$ with the norm of H and $\theta_{0m} \to \theta_0$ with the norm of G as $m \to \infty$ then

$$\begin{aligned} &|u_m(T)|^2 + \nu\int_0^T \|u_m(t)\|^2 dt \le |u_0|^2 + \frac{1}{\nu}\int_0^T \left\|f^1_{m-1}(t)\right\|^2_{V'} dt, \\ &|h_m(T)|^2 + \eta\int_0^T \|h_m(t)\|^2 dt \le |h_0|^2 + \frac{1}{\eta}\int_0^T \left\|f^2_{m-1}(t)\right\|^2_{V'} dt, \\ &|\theta_m(T)|^2 + \kappa\int_0^T \|\theta_m(t)\|^2 dt \le |\theta_0|^2 + \frac{1}{\kappa}\int_0^T \left\|f^3_{m-1}(t)\right\|^2_{W'} dt. \end{aligned} \qquad (27)$$

This shows that u_m and h_m remains in the bounded set of $L^2(0,T;V)$ and θ_m remains in a bounded set of $L^2(0,T;W)$.

From (24)-(26), we can see that

$$\begin{aligned}\frac{d}{dt}|u_m(t)|^2 &\le \frac{1}{\nu}\left\|f^1_{m-1}(t)\right\|^2_{V'},\\ \frac{d}{dt}|h_m(t)|^2 &\le \frac{1}{\eta}\left\|f^2_{m-1}(t)\right\|^2_{V'},\\ \frac{d}{dt}|\theta_m(t)|^2 &\le \frac{1}{\kappa}\left\|f^3_{m-1}(t)\right\|^2_{W'}.\end{aligned} \qquad (28)$$

Integrating the system (28) from 0 to s, we obtain

$$|u_m(s)|^2 \le |u_0|^2 + \frac{1}{\nu}\int_0^s \left\|f^1_{m-1}(t)\right\|^2_{V'} dt,$$

$$|h_m(s)|^2 \le |h_0|^2 + \frac{1}{\eta}\int_0^s \left\|f^2_{m-1}(t)\right\|^2_{V'} dt,$$

$$|\theta_m(T)|^2 \le |\theta_0|^2 + \frac{1}{\kappa}\int_0^s \left\|f^3_{m-1}(t)\right\|^2_{W'} dt.$$

Hence,

$$\sup_{0\le s\le T} |u_m(s)|^2 \le |u_0|^2 + \frac{1}{\nu}\int_0^s \left\|f^1_{m-1}(t)\right\|^2_{V'} dt, \qquad (29)$$

$$\sup_{0\le s\le T} |h_m(s)|^2 \le |h_0|^2 + \frac{1}{\eta}\int_0^s \left\|f^2_{m-1}(t)\right\|^2_{V'} dt, \qquad (30)$$

$$\sup_{0\le s\le T} |\theta_m(s)|^2 \le |\theta_0|^2 + \frac{1}{\kappa}\int_0^s \left\|f^3_{m-1}(t)\right\|^2_{W'} dt. \qquad (31)$$

The right hand sides of each of the inequalities (29)-(31) are finite and independent of m, therefore the sequence u_m and h_m remain in a bounded set of $L^\infty(0,T; H)$ and θ_m remains in a bounded set of $L^\infty(0,T; G)$.

Now, by using the above lemma we obtain the existence and uniqueness of the weak solutions of TEMHD problem.

Theorem *Let u_0, $h_0 \in H$ and $\theta_0 \in G$ then there exists the unique solution* (u, h, θ) *which satisfies problem* (2).

Proof: According to the result of above lemma, there exists an element u in $L^\infty(0, T; H)$ and a subsequence $u_{m'}$ such that $u_{m'}$ converges to u, for weak-star topology of $L^\infty(0, T; H)$. Also, $u_{m'}$ is in a bounded sequence in $L^2(0, T;V)$. Then, there exists $u^* \in L^2(0, T;V)$ and the sequence $u_{m''}$, the subsequence of $u_{m'}$ such that $u_{m''}$ converges to u* in weak topology of $L^2(0, T;V)$ (Friendman, 1982)

Hence, $u = u^* \in L^2(0, T;V) \cap L^\infty(0, T;H)$.

Similarly, we can show that $h \in L^2(0, T;V) \cap L^\infty(0, T;H)$ and $\theta \in L^2(0, T;W) \cap L^\infty(0, T;G)$. In order to pass the limit in equations (18)-(20) and their initial conditions, consider the scalar function $\psi(t)$ which is continuously differentiable on [0, T] and $\psi(T) = 0$. We multiply (18)-(20) by $\psi(t)$ and integrate with respect to t, from 0 to T and then taking limit $m = m' = m'' \to \infty$, we have

$$-\int_0^T (u(t), \psi'(t)x_j)dt + \nu\int_0^T ((u(t), \psi(t)x_j))dt = (u_0, x_j)\ \psi(0) + \int_0^T < f_1(t), x_j > \psi(t)dt, \quad (32)$$

$$-\int_0^T (h(t), \psi'(t)y_j)dt + \eta\int_0^T ((h(t), \psi(t)y_j))dt = (h_0, y_j)\ \psi(0) + \int_0^T < f_2(t), y_j > \psi(t)dt, \quad (33)$$

$$-\int_0^T (\theta(t), \psi'(t)z_j)dt + \kappa\int_0^T ((\theta(t), \psi(t)z_j))dt = (\theta_0, z_j)\ \psi(0) + \int_0^T < f_3(t), z_j > \psi(t)dt. \quad (34)$$

The equations (32)-(34) hold for each j and by continuity,

$$-\int_0^T (u(t), v_1)\psi'(t)dt + \nu\int_0^T ((u(t), v_1))\psi(t)dt = (u_0, v_1)\ \psi(0) + \int_0^T < f_1(t), v_1 > \psi(t)dt, \quad (35)$$

$$-\int_0^T (h(t), v_2)\psi'(t)dt + \eta\int_0^T ((h(t), v_2))\ \psi(t)dt = (h_0, v_2)\ \psi(0) + \int_0^T < f_2(t), v_2 > \psi(t)dt, \quad (36)$$

$$-\int_0^T (\theta(t), r)\psi'(t)dt + \kappa\int_0^T ((\theta(t), r))\,\psi(t)dt = (\theta_0, r)\ \psi(0) + \int_0^T < f_3(t), r > \psi(t)dt, \quad (37)$$

where v_1, v_2 and r are finite linear combinations of x_j's, y_j's and z_j's respectively.

Since each term of (35)-(37) depend linearly and continuously on v_1, v_2 and r respectively for each of the norm of V and W. Then the equations (35)-(37) are still valid.

Choosing $\psi(t) \in \mathcal{D}((0,T))$, we get

$$\frac{d}{dt}(u, v_1) + \nu((u, v_1)) = < f_1, v_1 >, \ \forall v_1 \in V, \quad (38)$$

$$\frac{d}{dt}(h, v_2) + \eta((h, v_2)) = < f_2, v_2 >, \ \forall v_2 \in V, \quad (39)$$

$$\frac{d}{dt}(\theta, r) + \kappa((\theta, r)) = < f_3, r >, \ \forall r \in W, \quad (40)$$

in distribution sense on (0, T).

The equation (38)-(40) imply the equations (11)-(13).

Also, we can easily check that the initial conditions (14) are satisfied.
So, we achieve the proof of the existence of weak solutions in linear case. Next, we will prove the uniqueness of the solutions in weak sense.

Assume that (u_1, h_1, θ_1) and (u_2, h_2, θ_2) be the solutions of the problem (1). Let $u = u_1 - u_2$, $h = h_1 - h_2$ and $\theta = \theta_1 - \theta_2$. Then u belongs to the same spaces of u_1, u_2 and also h and θ. So,

$$u' + \nu A_1 u = 0, \ u(0) = 0, \quad (41)$$

$$h' + \eta A_1 h = 0, \ h(0) = 0, \quad (42)$$

$$\theta' + \kappa A_2 \theta = 0, \ \theta(0) = 0. \quad (43)$$

Taking the scalar product of (41) with u(t), (42) with h(t) and (43) with θ(t), we get

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 = 0,$$

$$\frac{d}{dt}|h(t)|^2 + 2\eta\|h(t)\|^2 = 0,$$

$$\frac{d}{dt}|\theta(t)|^2 + 2\kappa\|\theta(t)\|^2 = 0.$$

Then $|u(t)|^2 \leq |u(0)|^2 = 0,\ |h(t)|^2 \leq |h(0)|^2 = 0,\ |\theta(t)|^2 \leq |\theta(0)|^2 = 0,\ \forall t \in [0, T].$

Hence, we can conclude the uniqueness of the solutions for each t in weak sense.

Conclusion

In this paper, first I have constructed the variational formulation of TEMHD problem using Faedo-Galarkin method and I have approximated the variational formulation of the problem. And then, I have proved that the solution extracted by Faedo-Galarkin method is weak convergent to the classical solution.

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