Hinthada University Research Journal, Vol. 5, No. 1, 2014

Some Facts from the Theory of Continued Fractions

Ko Ko Oo

Abstract

We presented the basic methods are carried to numerical examples with detailed steps of solution. Some theorems and corollaries' for converting the continued fraction to a simple fraction and vice versa are presented.

Key words: Continued fraction, simple fraction, numerical examples.

The Continued Fraction

Definition 1.1

An expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}} = \left[a_0;\ \frac{b_1}{a_1},\ \frac{b_2}{a_2},\ \frac{b_3}{a_3},\ \dots \right] \tag{1.1}$$

is called a *continued fraction* (Demidovich & Maron, 1970; Sastry, 1999).

The following abbreviated notation is also used for the continued fraction (1.1)

$$a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{|a_3} + \dots .$$

In the general case, the *elements* of a continued fraction a_0, a_k, b_k, ($k = 1, 2, \dots$) are real or complex numbers, or functions of one or more variables.

The fractions $a_0 = \frac{a_0}{1}$, $\frac{b_k}{a_k}$, ($k = 1, 2, \dots$) are called *components* of the continued fraction; (the *zeroth*, *first*, *second*, etc.), and the numbers of functions a_k and b_k, ($k \geq 1$) are called the *terms* of the k^{th} *component* (partial denominators or numerators). We will assume that $a_k \neq 0$.

If the continued fraction (1.1) contains a finite number of components *n*, not counting the *zeroth* one, it is called a *finite* or *n-component continued fraction* and is symbolized compactly as

$$\left[a_0;\ \frac{b_1}{a_1},\ \frac{b_2}{a_2},\ \frac{b_3}{a_3},\ \dots,\ \frac{b_n}{a_n} \right] = \left[a_0;\ \frac{b_k}{a_k} \right]_1^n \tag{1.2}$$

A finite continued fraction is identified with the corresponding common fraction obtained by performing the indicated operations. A continued fraction (1.1) having infinity of components is termed an *infinite continued fraction* and is defined as

$$\left[a_0;\ \frac{b_1}{a_1},\ \frac{b_2}{a_2},\ \frac{b_3}{a_3},\ \dots \right] = \left[a_0;\ \frac{b_k}{a_k} \right]_1^{\infty} \tag{1.3}$$

The continued fraction

Lecturer, Department of Mathematics, Hinthada University

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}} = [\, a_0;\ \frac{b_1}{a_1},\ \frac{b_2}{a_2},\ \frac{b_3}{a_3},\ \dots \,] \qquad (1.4)$$

where all partial numerators are equal to 1 is termed a *simple* (*standard*) *continued fraction*. The denominators of the components are called *partial quotients*. Note that in the theory of numbers, partial quotients are usually natural numbers (positive integers).

Converting a Continued Fraction to a Simple Fraction 1.2

Any finite continued fraction may be converted to a simple fraction. To do this, simply perform all the operations indicated by the continued fraction.

Converting a Simple Fraction to a Continued Fraction 1.3

Any positive rational number may be converted to a continued fraction with natural elements. Suppose we are given the fraction $\frac{p}{q}$.

Eliminating the integral part a_0, we have

$$\frac{p}{q} = a_0 + \frac{r_0}{q},$$

where r_0 is the remainder. If is a proper fraction then $a_0 = 0$ and $r_0 = p$.

Dividing the numerator and denominator of the fraction $\frac{r_0}{q}$ by r_0, we have

$$\frac{r_0}{q} = \frac{1}{q : r_0} = \frac{1}{a_1 + \frac{r_1}{r_0}},$$

where a_1 is an integral quotient and r_1 is the remainder left from dividing q by r_0.

Dividing the numerator and denominator of the fraction $\frac{r_1}{r_0}$ by r_1, we obtain

$$\frac{r_1}{r_0} = \frac{1}{r_0 : r_1} = \frac{1}{a_2 + \frac{r_2}{r_1}},$$

where a_2 is an integral quotient and r_2 is the remainder left from dividing r_0 by r_1.

The process may be continued in similar fashion.

Since $q > r_0 > r_1 > r_2 > \dots$ and r_i, ($i = 0, 1, 2, \dots$) are positive integers, we will finally have $r_n = 0$, or $\frac{r_{n-1}}{r_{n-2}} = \frac{1}{a_0 + 0}$.

Substituting the expressions of the fractions $\frac{r_i}{r_{i-1}}$, we get

$$\frac{p}{q} = a_0 + \frac{r_0}{q} = a_0 + \cfrac{1}{a_1 + \frac{r_1}{r_0}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{r_2}{r_1}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\dots}{\dots + \frac{1}{a_n}}}}.$$

Convergents 1.4

Suppose we have a terminating or nonterminating continued fraction

$$\left[a_0; \frac{b_k}{a_k} \right]_1^n \tag{1.5}$$

The simple fraction

$$\frac{P_k}{Q_k} = \left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots, \frac{b_k}{a_k} \right],$$ $k = 1, 2, 3, \ldots,$ where $k \le n$, is called the k^{th} *convergent* of the continued fraction (1.5).

We usually set

$$\frac{P_0}{Q_0} = \frac{a_0}{1}, \qquad \frac{P_{-1}}{Q_{-1}} = \frac{1}{0},$$

and for definiteness we assume that

$$P_0 = a_0, \ Q_0 = 1,$$

and $$P_{-1} = 1, \ Q_{-1} = 0. \tag{1.6}$$

Therefore

$$\frac{P_n}{Q_n} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{\cdots}{\cdots + \cfrac{b_n}{a_n}}}} + \ldots + \frac{b_n}{a_n}.$$

Then

$$c_1 = \frac{b_n}{a_n}, \qquad d_1 = a_{n-1} + c_1,$$

$$c_2 = \frac{b_{n-1}}{d_1}, \qquad d_2 = a_{n-2} + c_2,$$

$$\ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots$$

$$c_k = \frac{b_{n-k+1}}{d_{k-1}}, \qquad d_k = a_{n-k} + c_k,$$

$$\ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots$$

$$c_n = \frac{b_1}{d_{n-1}}, \qquad d_n = a_0 + c_n = \frac{P_n}{Q_n}$$

Theorem 1.5

The numbers P_k, Q_k, $(k = -1, 0, 1, 2, \ldots)$, determined from the relations

$$P_k = a_k P_{k-1} + b_k P_{k-2} \tag{1.7}$$

$$Q_k = a_k Q_{k-1} + b_k Q_{k-2} \tag{1.8}$$

where

$$P_{-1} = 1, \ Q_{-1} = 0, \ P_0 = a_0, \ Q_0 = 1 \tag{1.9}$$

are, respectively, the numerators and denominators of the convergent $\frac{P_k}{Q_k}$ of the continued fraction. We shall call such convergent *canonical*.

Proof : Let R_k, (k = 1, 2, …) be the successive convergent of the continued fraction (1.5).

It is required to prove that

$$R_k = \frac{P_k}{Q_k}, \quad (k = 1, 2, \dots).$$

We carry out the proof by *method of mathematical induction*.

When $k = 1$, we have, for the convergent R_1,

$$R_1 = a_0 + \frac{b_1}{a_1} = \frac{a_0 a_1 + b_1}{a_1}.$$

On the other hand, from relations (1.7) and (1.8), we get, taking into consideration (1.9),

$$P_1 = a_1 a_0 + b_1,$$

$$Q_1 = a_1 (1) + b_1 (0) = a_1.$$

Hence, $R_1 = \frac{P_1}{Q_1}$ and for $k = 1$ the assertion of the theorem holds.

Now let the theorem be true for all natural numbers not exceeding k. We will show that the theorem also holds true for the natural number (k+1).

From (1.7) and (1.8) we obtain

$$P_{k+1} = a_{k+1} P_k + b_{k+1} P_{k-1}, \tag{1.10}$$

$$Q_{k+1} = a_{k+1} Q_k + b_{k+1} Q_{k-1}.$$

By the induction hypothesis we have

$$R_k = \frac{P_k}{Q_k} = \frac{a_k P_{k-1} + b_k P_{k-2}}{a_k Q_{k-1} + b_k Q_{k-2}}.$$

By the law of formation of continued fraction (1.5), the convergent R_{k+1} is obtained from the convergent R_k by replacing the term a_k by the sum $a_k + \frac{b_{k+1}}{a_{k+1}}$.

Therefore

$$\begin{aligned} R_{k+1} &= \frac{(a_k + \frac{b_{k+1}}{a_{k+1}}) P_{k-1} + b_k P_{k-2}}{(a_k + \frac{b_{k+1}}{a_{k+1}}) Q_{k-1} + b_k Q_{k-2}} \\ &= \frac{a_{k+1}(a_k P_{k-1} + b_k P_{k-2}) + b_{k+1} P_{k-1}}{a_{k+1}(a_k Q_{k-1} + b_k Q_{k-2}) + b_{k+1} Q_{k-1}} \\ &= \frac{a_{k+1} P_k + b_{k+1} P_{k-1}}{a_{k+1} Q_k + b_{k+1} Q_{k-1}} \\ &= \frac{P_{k+1}}{Q_{k+1}}, \end{aligned}$$

which completes the proof.

Note 1.6

Since the terms of the convergent are not defined uniquely, in the general case, assert that the numerators and denominators of convergent of noncanonical type satisfy the equations (1.7) and (1.8).

Corollary 1.7

For the simple continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

the numerators and denominators of its convergent $\frac{p_k}{q_k}$ (k =1, 2, …) can be determined from the relations

$$p_k = a_k\, p_{k-1} + p_{k-2}\,,$$

$$q_k = a_k\, q_{k-1} + q_{k-2}\,, \qquad (1.11)$$

where we put $p_0 = a_0,\ p_{-1} = 1,$ and $q_0 = 1,\ q_{-1} = 0.$

Note 1.8

The following scheme is convergent for finding the terms of successive convergent from formulas (1.7) and (1.8).

k	-1	0	1	2	3	…
b_k	-	1	b_1	b_2	b_3	…
a_k	-	a_0	a_1	a_2	a_3	…
P_k	1	a_0	P_1	P_2	P_3	…
Q_k	0	1	Q_1	Q_2	Q_3	…

In this scheme, the row b_k is omitted for a continued fraction where $b_k = 1$ (k = 1, 2, …) and the formulas (1.11) hold.

Theorem 1.9

The formula

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = (-1)^{k-1}\,\frac{b_1 b_2 \dots b_k}{Q_{k-1}\, Q_k}\,, \quad (k \ge 1) \qquad (1.12)$$

holds true for two successive convergent $\frac{P_{k-1}}{Q_{k-1}}$ and $\frac{P_k}{Q_k}$ of the continued fraction.

<u>*Proof*</u> : We have

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = \frac{\Delta_k}{Q_{k-1}\, Q_k}\,, \qquad (1.13)$$

where

$$\Delta_k = \begin{vmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{vmatrix}.$$

Using relations (1.7) and (1.8), we get

$$\Delta_k = \begin{vmatrix} a_k P_{k-1} + b_k P_{k-2} & P_{k-1} \\ a_k Q_{k-1} + b_k Q_{k-2} & Q_{k-1} \end{vmatrix}$$

$$= b_k \begin{vmatrix} P_{k-2} & P_{k-1} \\ Q_{k-2} & Q_{k-1} \end{vmatrix} = -\mathrm{b_k}\, \Delta_{\mathrm{k-1}}.$$

From this we successively obtain

$$\Delta_\mathrm{k} = (-b_k)\ (-b_{k-1})\ \ldots\ (-b_1)\,\Delta_0$$
$$= (-1)^k\ b_1\, b_2 \ldots\ b_k\, \Delta_0$$

where

$$\Delta_0 = \begin{vmatrix} P_0 & P_{-1} \\ Q_0 & Q_{-1} \end{vmatrix} = \begin{vmatrix} a_0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Thus

$$\Delta_k = (-1)^{k-1}\ b_1\, b_2 \ldots\ b_k$$

and, consequently, on the basis of formula (1.13) we conclude that

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = (-1)^{k-1}\ b_1\, b_2 \ldots\ b_k\ \frac{1}{Q_{k-1}\, Q_k}\ .$$

Corollary 1.10

If $\dfrac{P_{k-1}}{Q_{k-1}}$ and $\dfrac{P_k}{Q_k}$ $(k \geq 1)$ are two successive convergent of the continued fraction, then

$$\Delta_k = P_k\, Q_{k-1} - P_{k-1}\, Q_k = (-1)^{k-1}\ b_1\, b_2 \ldots\ b_k\ .$$

Corollary 1.11

For two successive convergent $\dfrac{p_{k-1}}{q_{k-1}}$, $\dfrac{p_k}{q_k}$ $(k \geq 1)$ of a simple continued fraction, the following equation holds true:

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = (-1)^{k-1}\, \frac{1}{q_{k-1}\, q_k}\ . \tag{1.14}$$

Theorem 1.12

For two successive convergent of equal parity $\dfrac{P_{k-2}}{Q_{k-2}}$ and $\dfrac{P_k}{Q_k}$ $(k \geq 2)$ of the continued fraction, the relation

$$\frac{P_k}{Q_k} - \frac{P_{k-2}}{Q_{k-2}} = (-1)^k\,(\,b_1\, b_2 \ldots, b_{k-1},\ b_k)\ \frac{1}{Q_{k-2}\, Q_k}\,, \quad (1.15)$$

holds true.

Proof : We have $$\frac{P_k}{Q_k} - \frac{P_{k-2}}{Q_{k-2}} = \frac{d_k}{Q_{k-2}\ Q_k}, \quad (1.15)$$

where $$D_k = \begin{vmatrix} P_k & P_{k-2} \\ Q_k & Q_{k-2} \end{vmatrix}$$

whence, on the basis of the law of formation of convergents and on the basis of elementary properties of determinants, we obtain

$$D_k = \begin{vmatrix} a_k P_{k-1} + b_k P_{k-2} & P_{k-2} \\ a_k Q_{k-1} + b_k Q_{k-2} & Q_{k-2} \end{vmatrix}$$

$$= a_k \begin{vmatrix} P_{k-1} & P_{k-2} \\ Q_{k-1} & Q_{k-2} \end{vmatrix} = a_k \Delta_{k-1},$$

where Δ_k is the determinant considered in *Theorem* 1.9.

By the *corollary* 1.7 to *theorem* 1.5, we have

$$\Delta_{k-1} = (-1)^k\, b_1 b_2 \ldots b_{k-1},$$

whence $D_k = (-1)^k\, b_1 b_2 \ldots b_{k-1} a_k$.

Consequently we obtain formulas (1.15) by using relation (1.16).

Corollary 1.13

If $\frac{P_{k-2}}{Q_{k-2}}$ and $\frac{p_k}{q_k}$ are two successive convergents of the same parity of the simple continued fraction $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$ then we have the relation

$$\frac{p_k}{q_k} - \frac{P_{k-2}}{Q_{k-2}} = (-1)^k \frac{a_k}{q_{k-2} - q_k}. \quad (1.17)$$

Theorem 1.14

If all the elements of a finite continued fraction are positive, then its convergent of even order form a monotonic increasing sequence and the convergent of odd order form a monotonic decreasing sequence. Each convergent of even order is less than any convergent of odd order. The number α itself, which is expressed by the continued fraction, lies between two successive convergents.

Proof : Suppose we have the continued fraction

$$\alpha = \left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots, \frac{b_n}{a_n} \right]$$

with positive elements a_k and b_k and suppose $\frac{p_k}{q_k}$ (k = 0, 1, 2, … , n) are its successive canonical convergents.

Then obviously $P_k > 0$ and $Q_k > 0$.

We consider two cases:

(I) Let $k = 2m$ be an even number.

Then from (1.15), taking into consideration that $a_k > 0$ and $b_i > 0$, ($i = 1, 2, \ldots, k$), we have

$$\frac{P_{2m}}{Q_{2m}} - \frac{P_{2m-2}}{Q_{2m-2}} > 0.$$

Consequently,

$$\frac{P_{2m-2}}{Q_{2m-2}} < \frac{P_{2m}}{Q_{2m}}, \qquad (m = 1, 2, \ldots)$$

or $$\frac{P_0}{Q_0} < \frac{P_2}{Q_2} < \frac{P_4}{Q_4} < \ldots \tag{1.19}$$

(II) Let $k = 2m+1$ be odd number.

Then $(k-1)$ will be even, and from the same relation (1.15) we get

$$\frac{P_{2m-1}}{Q_{2m-1}} > \frac{P_{2m+1}}{Q_{2m+1}}$$

or $$\frac{P_1}{Q_1} > \frac{P_3}{Q_3} > \frac{P_5}{Q_5} > \ldots . \tag{1.20}$$

We have thus proved that even convergent form a monotonic increasing sequence and odd convergent form a monotonic decreasing sequence.

Furthermore, if in (1.17), we set $k = 2m$, we get

$$\frac{P_{2m-1}}{Q_{2m-1}} > \frac{P_{2m}}{Q_{2m}} \tag{1.21}$$

which is to say that any convergent of odd order is greater than the adjacent convergent of even order.

We conclude theorefrom that any convergent of odd order is greater than any convergent of even order.

Indeed, let $\dfrac{P_{2s-1}}{Q_{2s-1}}$ be any odd convergent.

If $s \le m$, then

$$\frac{P_{2s-1}}{Q_{2s-1}} \ge \frac{P_{2m-1}}{Q_{2m-1}} > \frac{P_{2m}}{Q_{2m}}$$

but if $s > m$, then

$$\frac{P_{2s-1}}{Q_{2s-1}} > \frac{P_{2m-1}}{Q_{2m-1}} > \frac{P_{2m}}{Q_{2m}}.$$

And so for any s and m we have

$$\frac{P_{2s-1}}{Q_{2s-1}} > \frac{P_{2m}}{Q_{2m}}. \quad (1.22)$$

Finally, from the law of formation of a continued fraction

$$\alpha = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}}.$$

We have the obvious relations $\alpha > \frac{P_0}{Q_0}, \ \alpha < \frac{P_1}{Q_1}, \ \alpha > \frac{P_2}{Q_2}, \ \dots$

Hence

$$\frac{P_k}{Q_k} < \alpha < \frac{P_{k+1}}{Q_{k+1}}, \text{ if } k \text{ is even} \quad (1.23)$$

and

$$\frac{P_k}{Q_k} > \alpha > \frac{P_{k+1}}{Q_{k+1}}, \text{ if } k \text{ is odd.} \quad (1.24)$$

For the last convergent, we will clearly have an equality on the right in place of the strict inequalities (1.23) and (1.24).

Acknowledgements

I would like to express my sincere gratitude to Dr. Aung Win, Pro-Rector, Hinthada University, for his kind permission to submit this paper. Special thanks are also due to Daw Aye Aye, Professor and Head of the Department of Mathematics, Hinthada University, for her encouragement and comments.

References

Demidovich, B. P., and I.A. Maron (1970) *Computational Mathematics*, Revised from Russian Edition, Mir Publishers, Moscow.

Sastry, S. S. (1999) *Introductory Methods of Numerical Analysis*, Third Edition, Prentice-Hall Private Limited.